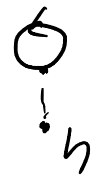
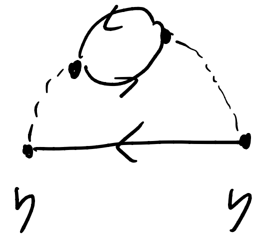
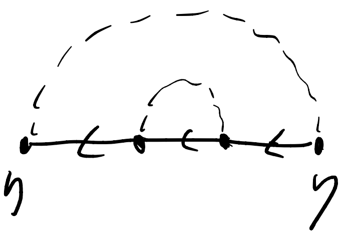
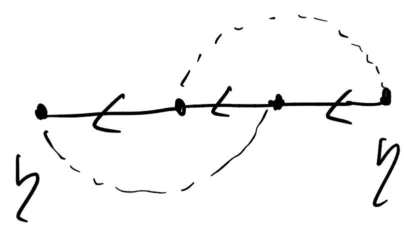
- Self-energy in Keldysh formalism:

$$-i\hat{\Sigma}_{\eta\eta'} = \delta_{\eta,\eta'} \Big($$  $$+$$  $$\Big) +$$  $$+$$

$$+$$  $$\delta_{\eta,\eta'} + \dots$$

↳ Structure with two external vertices of self-energy diagrams naturally makes $\hat{\Sigma}$ a matrix in Keldysh space

↳ $\check{\Sigma} = \hat{\tau}_3 \hat{\Sigma}$ and $\bar{\Sigma} = U \check{\Sigma} U^\dagger = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix}$ defined analogous to GFs.

↳ Diagrams such as  are <u>not</u> diagonal in Keldysh space, thus making $\hat{\Sigma}$ an entity with two time variables.

- Langreth rules for the real time formalism (A, B, C 2x2 Keldysh matrices)

↳ $(AB)^{R/A} = A^{R/A} B^{R/A}$; $(ABC)^{R/A} = A^{R/A} B^{R/A} C^{R/A}$ (clear from Larkin-Ovchinnikov rep.)

↳ $(AB)^{+-} = A^R B^{+-} + A^{+-} B^A$, $(ABC)^{+-} = A^R B^R C^{+-} + A^R B^{+-} C^A + A^{+-} B^A C^A$

(formally similar equations for "$+-$" → "$-+$" and "$+-$" → "K")

↳ Explicit application to the Keldysh component G^K in the Dyson eq.:

$$G^K = G_0^K + G_0^R \Sigma^R G^K + G_0^R \Sigma^K G^A + G_0^K \Sigma^A G^A \quad \text{(Keldysh-eq.)}$$

($G_0 \longleftrightarrow G$ in last 3 terms for second equation)

- Kadanoff-Baym equations

Using $(i\partial_{\tau_1} - H_0(1))\, G^0_\gamma(1,2) = \delta(\tau_1-\tau_2)\,\delta(x_1-x_2) = \delta(1,2)$, we derive from the Dyson equations:

$$\boxed{\begin{aligned} \left(i\partial_{\tau_1} - H_0(1)\right) G_\gamma(1,2) &= \delta(1,2) + \int d3\, \Sigma_\gamma(1,3)\, G_\gamma(3,2) \\ G_\gamma(1,2)\left(-i\partial_{\tau_2} - H_0(2)\right) &= \delta(1,2) + \int d3\, G_\gamma(1,3)\, \Sigma_\gamma(3,2) \end{aligned}}$$

(Kadanoff-Baym equations on the Keldysh contour)

↳ Using $\left(i\partial_{t_1} - H_0(1)\right) G_0^{+-}(1,2) = \underbrace{\left(i\partial_{t_1} - H_0(1)\right)}_{\equiv\, G_0^{-1}} G_0^{-+}(1,2) = 0$ as well as

$G_0^{-1}(1)\, G_0^R(1,2) = G_0^{-1}(1)\, G_0^A(1,2) = \delta(1,2)$, from the above

Langreth rules, we get

$$\boxed{\begin{aligned} G_0^{-1} G^K &= \Sigma^R G^K + \Sigma^K G^A \\ G^K G_0^{-1} &= G^R \Sigma^K + G^K \Sigma^A \end{aligned}}$$

(Kadanoff-Baym equations in real-time formalism)
(Formally identical equations of motion hold for G^{-+} and G^{+-})

↳ Integro-differential equations for two-time Green's functions.
See A.3.c for direct solution approaches.

↳ Common complementary approach (in particular interesting for semi-classical limit): Perform a Wigner-transform of G^{-+} and derive a quantum Boltzmann equation by means of gradient expansion (see, e.g., Mahan chapter 8.5, or Zagoskin chapter 3.5).

A.3.c: Conserving approximations

- Basic issue: Approximations to the self-energy may lead to violations of physical conservation laws obeyed by the equations of motion for the exact GF.

  ↳ Devise approximations that avoid unphysical effects by respecting the key symmetries.

- Conditions imposed on the self-energy by charge-conservation (continuity equation)

– Charge conservation for the exact equation of motion:

single-particle potential

$h_0(1) = -\frac{\nabla_1^2}{2m} + V(1)$ and $H_I = \frac{1}{2}\int d1 \int d2\, U(1-2)\, \psi_1^+ \psi_2^+ \psi_2 \psi_1$ gives

$$\left(i\frac{d}{d\tau_1} - h(1)\right) G_\gamma(1,2) = \delta(1,2) \pm i \int d3\, U(1-3)\, G^\gamma_{(2)}(1,3;2,3^+) \qquad \text{(M57)}$$

$$\left(-i\frac{d}{d\tau_2} - h(2)\right) G_\gamma(1,2) = \delta(1,2) \pm i\int d\bar{3}\, U(3-2)\, G^\gamma_{[2]}(1,3;2,3^+) \quad (MS2)$$

↳ With $\nabla_1^2-\nabla_2^2 = (\nabla_1+\nabla_2)(\nabla_1-\nabla_2)$ we get from (MS1) − (MS2):

$$\left[\left(i\frac{d}{d\tau_1}+\frac{i\,d}{d\tau_2}\right) + (\nabla_1+\nabla_2)\frac{(\nabla_1-\nabla_2)}{2m}\right] G_\gamma(1,2) = \pm i\int d^3\left[U(1-3)\, G^\gamma_{[2]}(1,3;2,3^+) - U(2-3)\, G^\gamma_{[2]}(1,3;2,3^+)\right]$$

↳ Equation for density $n(1)$ obtained by setting $2=1^+=(x_1,\tau_1^+)$

↳ $$\left[U(1-3)\, G^\gamma_{[2]}(1,3;2,3^+) - U(2-3)\, G^\gamma_{[2]}(1,3;2,3^+)\right]\Big|_{2=1^+} = 0 \quad (MS3)$$

↳ $$\nabla_1\cdot\underbrace{\left(\frac{\nabla_1-\nabla_2}{2m}\, G_\gamma(1,2)\right)\Big|_{2=1^+}}_{=\pm j(1)} + \frac{d}{d\tau_1}\underbrace{i\, G_\gamma(1,2)\Big|_{2=1^+}}_{=\pm n(1)} = 0$$

$$\Longleftrightarrow \quad \boxed{\nabla_1\cdot j(1) + \frac{d}{d\tau_1} n(1) = 0} \quad \text{(continuity equation)}$$