$$\nabla_1 \cdot j(1) + \frac{d}{d\tau_1} n(1) = 0 \qquad \text{(continuity equation)}$$

$$j(x,\tau) = \frac{1}{2mi} \frac{\mathrm{Tr}\left[e^{-\beta H}\left(\psi_H^\dagger(x,\tau)\left(\nabla\psi_H(x,\tau)\right) - \left(\nabla\psi_H^\dagger(x,\tau)\right)\psi_H(x,\tau)\right)\right]}{\mathrm{Tr}\left[e^{-\beta H}\right]} =$$

$$= \pm \left(\frac{\nabla - \nabla'}{2m} G_\gamma(x,\tau;x',\tau^+)\right)\Big|_{x'=x}$$

- For an approximate self-energy Σ, the trivial equation

$$\left[U(1-3)\, G^\gamma_{(2)}(1,3;2,3^+) - U(2-3)\, G^\gamma_{(2)}(1,3;2,3^+)\right]\Big|_{2=1^+} = 0$$

becomes a nontrivial condition on the self-energy:

$$\int d3 \left[\Sigma_\gamma(1,3)\, G_3(3,1^+) - G_\gamma(1,3)\, \Sigma_\gamma(3,1^+) \right] = 0 .$$

- Similar conditions are imposed on Σ by (angular) momentum- and energy-conservation (see 8.3–8.6 and 9.2 in Stefanucci an v. Leeuwen)

$\hookrightarrow$ $$\int d3 \int dx_1 \left[\Sigma_\gamma(1,3)\, \nabla_1 G_\gamma(3,1^+) - G_\gamma(1,3)\, \nabla_1 \Sigma_\gamma(3,1^+) \right] = 0 \quad \text{(momentum)}$$

$\hookrightarrow$ $$\int d3 \int dx_1 \left(x_1 \times \left[\Sigma_\gamma(1,3)\, \nabla_1 G_\gamma(3,1^+) - G_\gamma(1,3)\, \nabla_1 \Sigma_\gamma(3,1^+) \right] \right) = 0 \quad \text{(angular momentum)}$$

$\hookrightarrow$ $$\int d3 \int dx_1 \left\{ \frac{1}{4} \frac{d}{dz_1} \left[\Sigma_\gamma(1,3) G_\gamma(3,1) + G_\gamma(1,3) \Sigma_\gamma(3,1) \right] - \left[\Sigma_\gamma(1,3) \left(\frac{d}{dz_1} G_\gamma(3,1) \right) + \left(\frac{d}{dz_1} G_\gamma(1,3) \right) \Sigma_\gamma(3,1) \right] \right\} = 0$$

(energy).

- The Φ-functional is a tool to derive self-energies that manifestly conserve the constants of motion of the system.

- Basic idea: Consider a functional $\Phi[G]$ of the full GF G_γ such that $\Sigma_\gamma(1,2) = \frac{\delta\Phi[G_\gamma]}{\delta G_\gamma(2,1^+)}$ (functional derivative)

- If the functional Φ is symmetric under an infinitesimal variation δG_γ, then
$$0 = \delta\Phi = \int d1 \int d2 \; \Sigma_\gamma(1,2)\, \delta G_\gamma(2,1^+)$$
(note similarity to conditions on Σ above)

⇝ If Φ obeys symmetries such as charge-, (angular) momentum- .... conservation, the Σ obtained as its functional derivative also respects these conservation laws. (see G. Baym Phys. Rev. 127, 1391 (1962)).

↳ Concrete example of charge conservation corresponding to U(1) gauge symmetry: $\Psi(1) \to e^{-i\chi(1)}\Psi(1)$; $G(1,2,\chi) = e^{-i\chi(1)}\, G(1,2,0)\, e^{i\chi(2)}$.

[index γ dropped for brevity]

$$0 = \delta \Phi[G] = \Phi[G+\delta G] - \Phi[G] = \int d1 \int d2 \, \frac{\delta \Phi[G]}{\delta G(1,2)} \, \delta G(1,2,x) \Big|_{x=0} =$$

$$= \int d1 \int d2 \, \Sigma(2,1) \left[\frac{\partial G(1,2,x)}{\partial x(1)} \Big|_{x=0} \delta x(1) + \frac{\partial G(1,2,x)}{\partial x(2)} \Big|_{x=0} \delta x(2) \right] =$$

[$G(1,2,x=0) = G(1,2)$]

$$= -i \int d1 \int d2 \left[\Sigma(2,1) G(1,2) \, \delta x(1) - \Sigma(2,1) G(1,2) \, \delta x(2) \right] =$$

[$1 \leftrightarrow 2$ in second term]

[arbitrary variation]

$$= -i \int d1 \left\{ \int d2 \left[\Sigma(2,1) G(1,2) - G(2,1) \Sigma(1,2) \right] \right\} \delta x(1)$$

$$\leadsto \int d2 \left[G(1,2) \Sigma(2,1) - \Sigma(1,2) G(2,1) \right] = 0 \quad \checkmark$$

↳ Similar derivations yield the corresponding conditions on Σ for (angular) momentum- and energy-conservation.

- Physical meaning and form of the Φ-functional

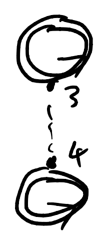
  - Key requirements on Φ:

    (i) Φ obeys physical conservation laws
    (ii) Σ is obtained from Φ by functional derivative w.r.t. G

  - Observation: Vacuum diagrams of the problem at hand obey (i) as they represent scattering processes, and functional derivatives w.r.t. G for G-skeleton diagrams yield irreducible self-energy diagrams

    (graphically $\dfrac{\delta \Phi}{\delta G(1,2)}$ corresponds to removing $G(1,2)$ from the diagram)

  - Lowest order examples:



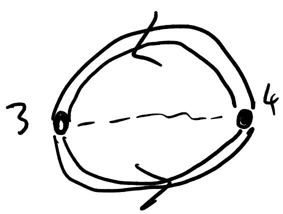
(Hartree)

$$\Phi_H[G] = \int d3 \int d4 \left(-iG(3,3^+)\right)\left(-iU(3-4)\right)\left(-iG(4,4^+)\right)$$

$$\rightsquigarrow \quad \Sigma_H(1,2) = \frac{\delta \Phi_H[G]}{\delta G(2,1)} = \overset{3\leftrightarrow 4}{2i} \int d3 \int d4\, \delta(1-3)\,\delta(2-3)\, U(3-4)\, G(4,4^+) =$$

$$= 2i\,\delta(1-2) \int d3\, U(2-3)\, G(3,3^+)$$

$\rightsquigarrow$ $\Sigma_H(1,2)$ is indeed the Hartree self-energy up to a factor $\underset{\text{Fermi}}{\pm}\frac{1}{2}$.



(Fock)

$$\simeq \quad \phi_F[G] = \overset{\text{one Fermi-loop}}{\mp} \int d3 \int d4\, iG(3,4)\,\big(-iU(3-4)\big)\, iG(4,3)$$

$$\Sigma_F(1,2) = \frac{\delta \phi[G]}{\delta G(2,1)} = \mp \overset{3\leftrightarrow 4}{2i} \int d3 \int d4\, \delta(3,2)\,\delta(4,1)\, U(3-4)\, G(4,3)$$

$$= \mp U(2-1)\, G(1,2)\, 2i$$

$\rightsquigarrow$ $\Sigma_F(1,2)$ is the Fock self-energy up to a prefactor of $\underset{\text{Fermi}}{\pm}\frac{1}{2}$.

↳ Since linear combinations of scattering processes that individually preserve symmetries are still symmetric, we may write

$$\Phi_{HF} = \pm\frac{1}{2}\Phi_{H} \pm \frac{1}{2}\Phi_{F} \quad \text{which yields} \quad \frac{\delta\Phi_{HF}[G]}{\delta G(2,1)} = \Sigma_{HF}(1,2)$$

with the correct Hartree-Fock self-energy Σ_{HF}.