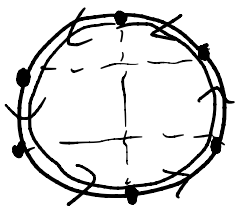
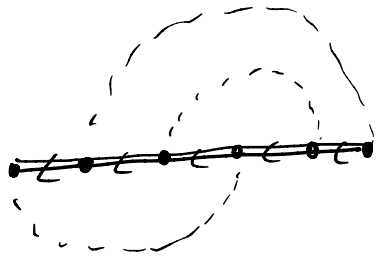
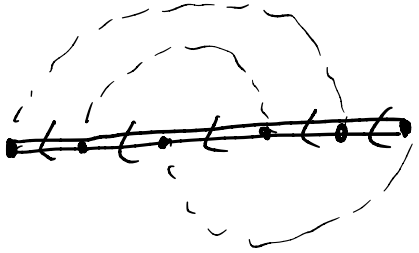
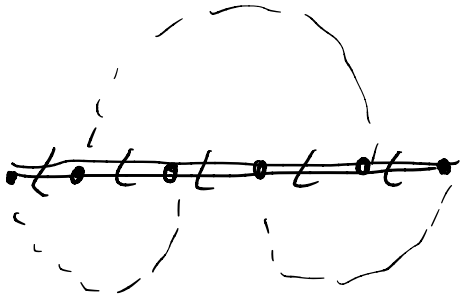
- Non-trivial higher order example of Φ-derivable combinations of self-energy diagrams:

Φ =  $\rightsquigarrow \Sigma(1,2) = \frac{\delta \Phi}{\delta G(2,1^+)} \simeq$ 2 x 

+ 2x  + 2x  .

$\rightsquigarrow$ Conserving approximation must take all three diagrams into account with correct relative prefactor.

- Conserving approximations with <u>dressed</u> interaction possible?

Yes (see 11.8 Stefanucci and van Leeuwen):

Ψ – functional consists of vacuum diagrams with dressed (effective) interaction that yield W-skeleton diagrams on removing one GF.

$$\left(\Sigma - \Sigma_{\text{Hartree}}\right)(1,2) = \frac{\delta \Psi[G,W]}{\delta G(2,1^+)}$$

$$\Pi_{\text{irred.}}(1,2) = \pm 2 \left.\frac{\delta \Psi[G,W]}{\delta W(1,2)}\right|_{\text{symmetric in } 1 \leftrightarrow 2}$$

- Self-energies that can be derived from a functional Φ obeying (i), (ii) are called Φ-derivable approximations that are by construction symmetry preserving.
- Recipe for higher conserving approximations than Hartree-Fock
  - $\Phi[G]$ is the sum of all (topologically inequivalent) G-skeleton vacuum diagrams weighted by a symmetry factor N_S that accounts for multiple counting of equivalent diagrams. It can be shown that $N_S = \frac{1}{2n}$ where n is the order in the interaction.

Formally $$\boxed{\Phi[G] = \sum_{n=1}^{\infty} \frac{1}{2n} \int d1 \int d2 \, \Sigma_S^{(n)}(1,2) G(2,1)}$$

- Comments:
  ↳ Diagrammatics of Φ is similar to those of $\Omega = -T \log Z$,

combinatorial prefactors do not cancel out as elegantly as in the series for G (thus the need for N_s)
(see 11.3 in Stefanucci and van Leeuwen).

↳ $\Phi[G]$ actually explicitly depends on $U(1^+-2)$, but we surpressed this dependence for derivatives w.r.t. G.

- Integrating the Kadanoff-Baym equations (KBE) for low-order conserving approximations.

– Dynamical Hartree-Fock

$$\left(i\frac{d}{d\bar{t}_1} - H_0(1)\right) G_\gamma(1,2) = \delta(1,2) + \int d3\, \Sigma_{HF}(1,3)\, G_\gamma(3,2)$$

$$\Sigma_{HF}(1,2) = \underbrace{\delta(1,2)\left(\pm i \int d3\, U(2,3)\, G_\gamma(3,3^+)\right)}_{\delta(1,2)\, V_H(1)}$$

Σ local in space and time

$$+ \quad \mp i \underbrace{U(1,2)\, G_\gamma(1,2)}$$

$$\mp i \delta(\tau_1 - \tau_2)\, U(x_1 - x_2)\, G_\gamma(x_1, \tau_1; x_2, \tau_1^+)$$

Σ local in time but non-local in space

matrices in space

$$\hookrightarrow \left(i \frac{d}{d\tau_1} \underbrace{- H_0(\tau_1) - V_{HF}(\tau_1)}_{-\hat{H}_{HF}(\tau_1)} \right) G_\gamma(\tau_1, \tau_2) = \delta(\tau_1 - \tau_2)\, \mathbb{1}_x$$

with $$\hat{V}_{HF}(\tau_1) = \int dx_1 \int dx_2 \; |x_1\rangle \Big(\delta(x_1 - x_2) V_H(x_1, x_2) \mp i U(x_1 - x_2)\, G_\gamma(x_1, \tau_1; x_2, \tau_1^+) \Big) \langle x_2|$$

$\hookrightarrow$ KBE formally equivalent to Schrödinger equation with time-dependent Hamiltonian $\hat{H}_{HF}(\tau_1)$

$\rightsquigarrow$ $\hat{H}_{HF}$ local for local interactions such as Hubbard interactions.

$\rightsquigarrow$ H_{HF} local in momentum space for translation invariant systems.

$\rightsquigarrow$ Branches of Keldysh contour remain uncoupled, and contour structure is irrelevant

↳ Formal solution: HF time evolution operator

$$\hat{U}_{HF}(t,t_0) = \mathcal{T} e^{-i\int_{t_0}^{t} H_{HF}(t')\,dt'}$$

$\rightsquigarrow$ Self-consistent solution by time-evolving the equal-time GF:

$$\rho(t) \equiv G_1^{-+}(t,t) = U_{HF}(t,t_0)\, G_1^{-+}(t_0,t_0)\, U^{\dagger}(t,t_0)$$

one-body density matrix

$\rightsquigarrow$ Divide time into small steps δt and iteratively update $H_{HF}(t') \to H_{HF}(t'+\delta t)$ at every step.