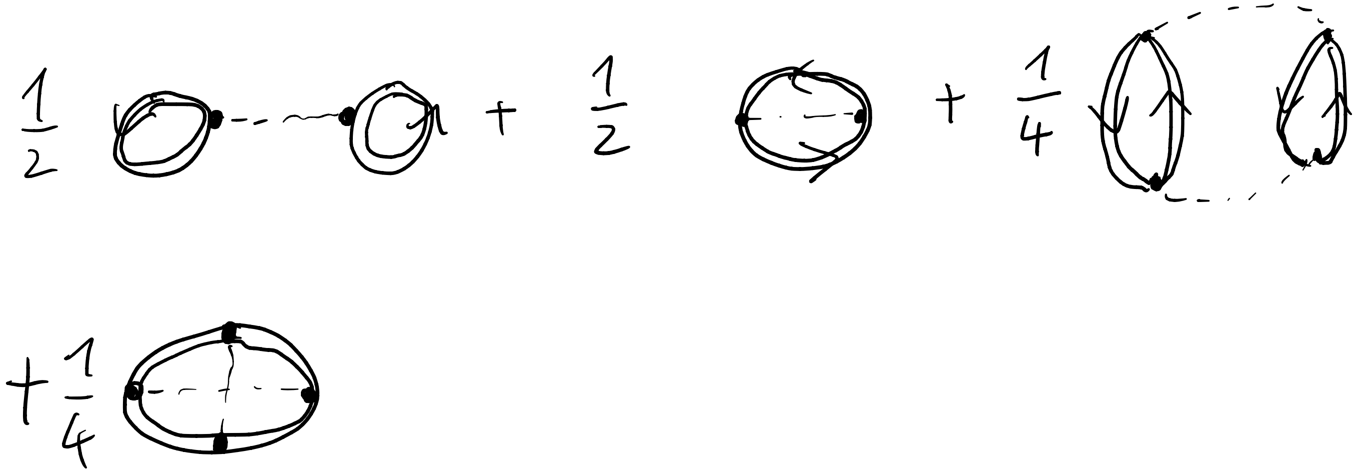
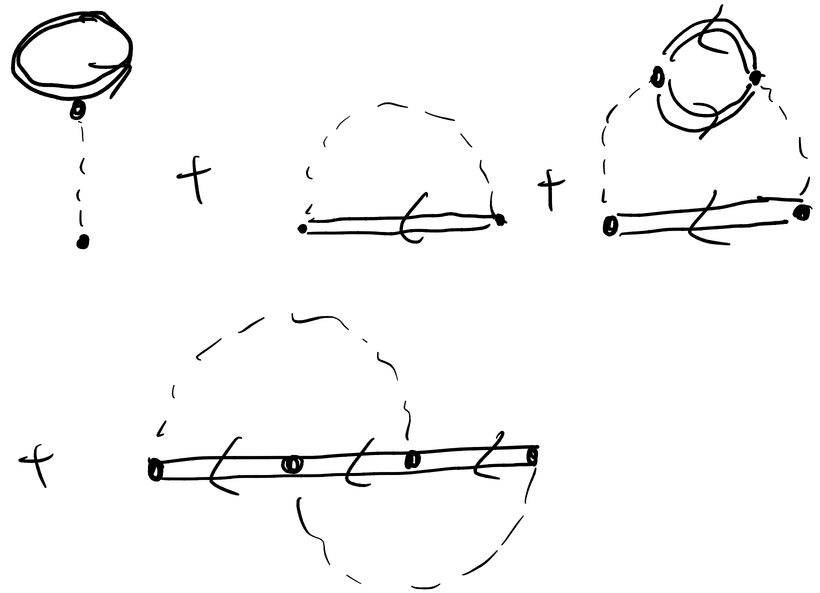
- Self-consistent second Born approximation

$$\pm \Phi_{2BA} =$$



$$\rightsquigarrow \Sigma_{2BA}(1,2) = \frac{\delta \Phi_{2BA}}{\delta G(2,1)} =$$



↳ By contrast to the HF case, the self-energy is non-local in time (non-diagonal in Keldysh space)

$\rightsquigarrow$ KBE must be integrated with the full 2-time structure.

(See Section 16.3 in Stefanucci, van Leeuwen for details)
(Discussion for fermions)

↳ KBEs to be solved (H_0 can be understood as H_{HF})

$$\left(i\frac{d}{dt_1} - H_0(t_1)\right) G^{+-}(t_1,t_2) = \left[\Sigma^R G^{+-} + \Sigma^{+-} G^A\right](t_1,t_2)$$

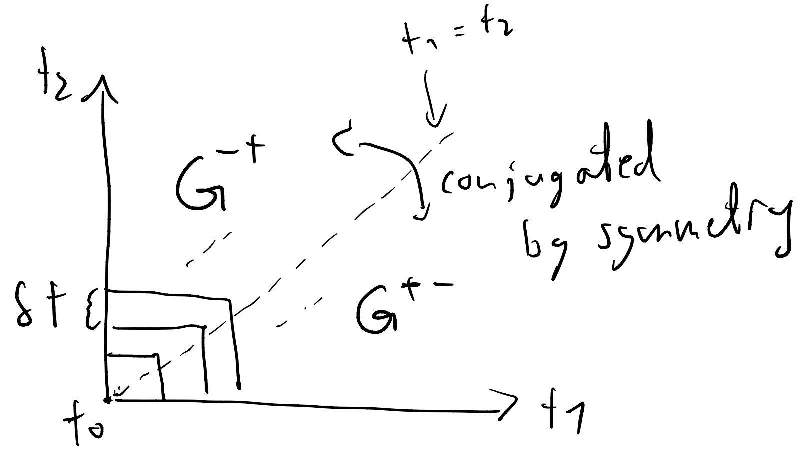
$$G^{-+}(t_1,t_2)\left(-i\frac{d}{dt_2} - H_0(t_2)\right) = \left[G^R \Sigma^{-+} + G^{-+} \Sigma^A\right](t_1,t_2)$$

along with the equal time equation

$$i\frac{d}{dt} G^{-+}(t,t) - \left[H_0(t), G^{-+}(t,t)\right] = -\left[G^R\Sigma^{-+} + G^{-+}\Sigma^A\right](t,t) + h.c.,$$

where for equal times $G^{-+}(t,t) = -i\mathbb{1} + G^{+-}(t,t)$, and generally $G^{\substack{-+\\+-}}(1,2) = -G^{\substack{+-\\-+}}(2,1)^{\dagger}$

⇝ Solution by time-stepping starting at (t_0,t_0) and extending into the two-time rectangle:



$\hookrightarrow$ Problem: KBEs at (t_1, t_2) involve "collision integrals" that at first glance may seem to require knowledge of GFs at all times.

$\rightsquigarrow$ Resolved by causal structure of collision integrals:

$$I^{+-}(t_1,t_2) = \left[\Sigma^R G^{+-} + \Sigma^{+-} G^A\right](t_1,t_2) = \int_{t_0}^{\infty} dt_3 \left[\Sigma^R(t_1,t_3)\, G^{+-}(t_3,t_2) + \Sigma^{+-}(t_1,t_3)\, G^A(t_3,t_2)\right] = \int_{t_0}^{t_1} \Sigma^R(t_1,t_3)\, G^{+-}(t_3,t_2)\, dt_3 + \int_{t_0}^{t_2} \Sigma^{+-}(t_1,t_3)\, G^A(t_3,t_2)\, dt_3$$

$$\boxed{\begin{aligned} &\Sigma^R \sim \theta(t_1 - t_3) \\ &G^A \sim \theta(t_2 - t_3) \end{aligned}}$$

Hence, the self-energies and GFs only need to be known up to time $T := \max(t_1, t_2)$, which is consistent with the strategy of integrating the KBEs on the two-time rectangle starting from (t_0, t_0).

↳ Problem: Computation of $\Sigma(t_1, t_2)$ from the considered diagrams involves further time-integrals of GFs at internal vertices.

→ Explicit example of second Born approximation:

$$\Sigma^{\gamma}_{2c}(1,2) = \left[\Sigma^{\gamma}_{2BA} - \Sigma^{\gamma}_{HF}\right](1,2) = \int d3 \int d4\, U(1-3)\, U(2-4) \left[G_\gamma(1,2)\right.$$

$$\left. G_\gamma(3,4)\, G_\gamma(4,3) - G_\gamma(1,4)\, G_\gamma(4,3)\, G_\gamma(3,2) \right]$$

For the "+-" and "-+" components, assuming $U(1-3) = \delta(\tau_1 - \tau_3)\, \tilde{U}(\tau_1)$, i.e. for interactions that may depend on time, but are local in time):

$$\Sigma_{2c}^{+-}(t_1,t_2) = \int dx_3 \int dx_3\, \tilde{U}(t_1, x_1-x_3)\, \tilde{U}(t_2, x_2-x_4) \Big[G^{+-}(t_1,x_1;t_2,x_2)\, G^{+-}(t_1,x_3;t_2,x_4)\, G^{-+}(t_2,x_4;t_1,x_3) \; + $$

T branch of all time arguments fixed by δ-functions

$$ - \; G^{+-}(t_1,x_1;t_2,x_2)\, G^{-+}(t_2,x_4;t_1,x_3)\, G^{+-}(t_1,x_3,t_2,x_2) \Big]$$

similar for other components

$\rightsquigarrow$ To compute $\Sigma_{2c}^{\pm\mp}$ with time-arguments up to $T = \max(t_1,t_2)$, we only need GFs with arguments up to T, thus enabling the explicit solution of the KBEs on the two-time rectangle by time stepping.

A.3.d: Non-equilibrium perturbation theory including initial correlations

- Back to the basic definition of the (causal) GF:

$$G_c(x,t;x',t') = -i\langle T(\Psi_H(x,t)\Psi_H^+(x',t'))\rangle \overset{\text{[drop spatial indices]}}{=}$$

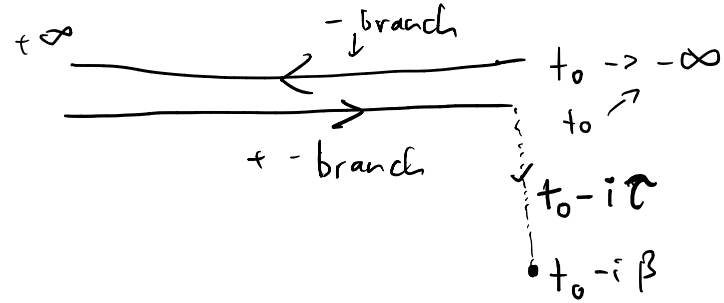
$$= -\frac{i}{Z}\,\mathrm{Tr}\left(e^{-\beta H}\,T_K\left(e^{-i\int_\gamma d\bar{\tau}\,V_I(\bar{\tau})}\,\Psi(t^-)\Psi^+(t'^-)\right)\right) =$$

2-branch Keldysh contour

[see also A.2 on Matsubara formalism]

$$= -i\,\frac{\langle T_{K_3}\left(e^{-i\int_{\gamma_3} dz\,V_I(z)}\,\Psi(t^-)\Psi(t'^-)\right)\rangle_0}{\langle S_M(\beta,0)\rangle_0}$$

with the new contour $\gamma_3 \simeq$



$$\hookrightarrow G_{\gamma_3}(z, z') = -i \frac{\left\langle T_{K_3}\left(e^{-i\int_{\gamma_3} V_I(\tilde{z})} \psi(z)\psi^\dagger(z') \right) \right\rangle_0}{\langle S_M(\beta, 0)\rangle_0}$$

has time-arguments $z, z' \in \gamma_3$.

- In the diagrammatic perturbation theory, besides G^M, G^R, G^A, G^K, two new GFs must be considered:

$$G^{\ulcorner}(\tau, t) := G_{\gamma_3}(t_0 - i\tau, t^{\pm}) \quad ; \quad G^{\urcorner}(t, \tau) = G_{\gamma_3}(t^{\pm}, t_0 - i\tau).$$

identical for $t^{\pm}$
(differs by uninterrupted $U^\dagger U = \mathbb{1}$)