- More quantitative analysis of 1D case

— Consider $\varepsilon(k) = \dfrac{k^2 - k_F^2}{2m}$; i.e. $E_k(q) \equiv \varepsilon(k+q) - \varepsilon(k) = \dfrac{2kq + q^2}{2m}$,

and $k \in [k_F - q, k_F]$ such that k is occupied and $k+q$ empty,

($q > 0$, $k > 0$ branch)

$$\hookrightarrow \underbrace{E(q)}_{\text{"average energy"}} \equiv \frac{\overset{k=k_F}{\max_k}\left(E_k(q)\right) + \overset{k=k_F-q}{\min_k}\left(E_k(q)\right)}{2} = \frac{1}{4m}\left(2k_F q + q^2 + 2(k_F - q)q + q^2\right) =$$

$$= \frac{k_F}{m} q = \underbrace{v_F}_{\text{Fermi-velocity}} q \,.$$

$$\hookrightarrow \delta E(q) \equiv \max_k\left(E_k(q)\right) - \min_k\left(E_k(q)\right) = \frac{1}{2m}\left(2k_F q + q^2 - 2(k_F - q) - q^2\right) =$$

$$\frac{q^2}{m} = \frac{E(q)^2}{m v_F^2} \,.$$

- Generalize to generic expansion of $\varepsilon(k)$ around k_F ($k>0$ branch)

$$\varepsilon(k) = v_F\,(k-k_F) + \frac{\lambda}{2}(k-k_F)^2 + \dots$$

$$\rightsquigarrow\ E(q) = v_F\, q\ ,\quad \delta E(q) = \lambda q^2 = \frac{\lambda}{v_F^2} E(q)^2 .$$

↳ $E(q)$ only depends on q and defines a quasi-particle dispersion for particle-hole excitations

↳ $\delta E(q)$ is a higher order (band-bending) effect that may be neglected within the validity regime of the linearized dispersion (i.e. close to E_F).

- Particle-hole excitations created by $\sum_k a^\dagger_{k+q} a_k$, which even for fermionic a_k is a bosonic operator. This observation is at the heart of the bosononization technique discussed in part B.2

B.2.a Phenomenological Bosonization: Quick and dirty

- Basic ingredients: (i) Density as a bilinear in field operators is bosonic
  (ii) 1D geometry allows for ordering relation of particles from left ($x \to -\infty$) to right ($x \to +\infty$)

- Bosonic counting field:

$$\vartheta(x) = \pi \int_x^\infty dy\, n(y) \rightsquigarrow n(x) = -\frac{1}{\pi} \partial_x \vartheta(x)$$

↳ $\vartheta(x)$ counts particles to the right of position x.

↳ Creating a particle at position x' increases $\vartheta(x)$ by π for all $x < x'$.

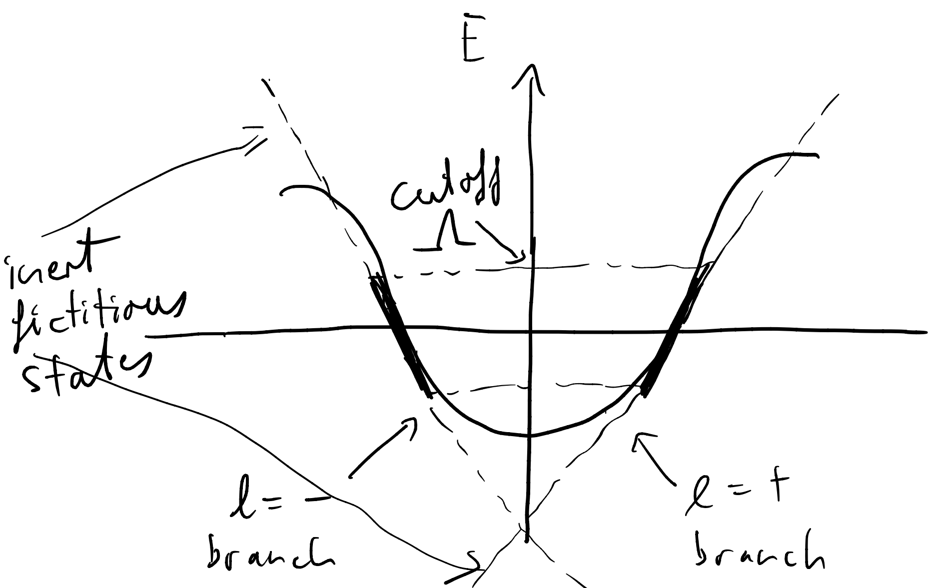
↳ Introducing the "conjugate momentum" field $\Pi(x)$ which satisfies $[\rho(x), \Pi(y)] = i\delta(x-y)$, we see that the operator $e^{i\lambda \int_{-\infty}^{x'} dy\, \Pi(y)} \equiv \tilde{\Psi}^+(x')$ precisely satisfies

$$[\tilde{\Psi}^+(x'), \rho(x)] = \begin{cases} 0 & ; x > x' \\ \lambda \tilde{\Psi}^+(x') & ; x < x' \end{cases}$$

, i.e it achieves the desired shift of adding a particle at x'.

$\rightsquigarrow$ Fermionic creation-operators should be similar to $\tilde{\Psi}^+$, up to a factor that commutes with $\rho(x)$.

- Free fermionic model: Tomonaga-Luttinger Hamiltonian



subtract "vacuum" energy

$$H_0^F = \sum_{l=\pm} \sum_k \varepsilon_{l,k} \left(a_{l,k}^+ a_{l,k} - \langle a_{l,k}^+ a_{l,k} \rangle_0 \right)$$

$$\varepsilon_{l,k} = \pm v_F (\underbrace{k \mp k_F}_{q}), \quad l = \pm.$$

– Decomposition into right- and left-movers:

$$\Psi(x) = e^{ik_F x}\,\Psi_+(x) + e^{-ik_F x}\,\Psi_-(x) \, ;$$

$$\Psi_+(x) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi}\, e^{iqx}\, a_{+,q+k_F} \quad , \quad \Psi_-(x) = \int_{-\infty}^{+\infty} \frac{dq}{2\pi}\, e^{iqx}\, a_{-,q-k_F} \, .$$

$$\rightsquigarrow H_0^F = v_F \int dx \left(\Psi_+^\dagger(x) \frac{\partial_x}{i} \Psi_+(x) - \Psi_-^\dagger(x) \frac{\partial_x}{i} \Psi_-(x) \right) - \langle \ldots \rangle_0$$

$\rightsquigarrow$ excitation energies $\varepsilon_q = v_F |q|$

- Free bosonic model

$$H_0^B = \frac{1}{2} v \int dx \left[\Pi(x)^2 + (\partial_x \phi(x))^2 \right] \quad \text{with} \quad [\Pi(x), \phi(y)] = -i\,\delta(x-y)$$

- From $\partial_t \phi(x,t) = -i\left[\phi(x,t), H_0^B\right] = v\,\Pi(x,t)$, we get $\boxed{\Pi = \frac{1}{v}\partial_t \phi}$

- Fourier decomposition $\quad \phi(x) = \int \frac{dk}{2\pi} \sqrt{\frac{v}{2\omega(k)}} \left(b(k)\, e^{ikx} + b^\dagger(k)\, e^{-ikx} \right)$

$\left[\omega(k) = v|k| , \right.$

$\left. [b(k), b^\dagger(k')] = 2\pi\,\delta(k-k') \right]$

$$\Pi(x) = \int \frac{dk}{2\pi} \sqrt{\frac{\omega(k)}{2v}} \left[-i\, b(k)\, e^{ikx} + i\, b^\dagger(k)\, e^{-ikx} \right].$$

- Check: $[\phi(x), \Pi(y)] = \frac{1}{2} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \Big(i e^{i(kx - yk')} \underbrace{[b(k), b^\dagger(k')]}_{= 2\pi\delta(k-k')}$

$- i e^{-i(kx - k'y)} \underbrace{[b^\dagger(k), b(k')]}_{= -2\pi\delta(k-k')} =$

$$= \frac{i}{2} \left(\int \frac{dk}{2\pi} e^{ik(x-y)} + \int \frac{dk}{2\pi} e^{-ik(x-y)} \right) = i\,\delta(x-y)$$ ✓

- $H_0^B =$ .... $= \int \frac{dk}{2\pi} \omega(k)\, b^\dagger(k)\, b(k)$, with $\omega(k) = v|k|$ ✓

plug in ϕ and π in terms of b and $b^\dagger$, and simplify

$b(k,t) = e^{-i\omega(k)t} b(k)$

$$\rightsquigarrow \phi(x,t) = \int \frac{dk}{2\pi} \sqrt{\frac{v}{2\omega(k)}} \left[b(k)\, e^{i(kx - v|k|t)} + b^\dagger(k)\, e^{-i(kx - v|k|t)} \right]$$

- Decomposition into right- and left-moving part:

$$\varphi(x,t) = \phi_+(x-vt) + \phi_-(x+vt) \quad \text{with}$$

$$\phi_+(x-vt) = \int_{k>0} \frac{dk}{2\pi} \frac{1}{\sqrt{2k}} \left[b(k) e^{ik(x-vt)} + b^\dagger(k) e^{-ik(x-vt)} \right]$$

$$\phi_-(x+vt) = \int_{k<0} \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left[b(k) e^{ik(x+vt)} + b^\dagger(k) e^{-ik(x+vt)} \right].$$