- Defined dual field by $\partial_x \vartheta = -\Pi = -\frac{1}{v}\partial_t \varphi$

↳ From $\partial_t \varphi(x,t) = \partial_t \phi_+(x-vt) + \partial_t \phi_-(x+vt) = -v\,\partial_x \phi_+(x-vt)$
$+ v\,\partial_x \phi_-(x-vt)$, we get

$$\partial_x \vartheta = \partial_x(\phi_+ - \phi_-) \rightsquigarrow \boxed{\vartheta = \phi_+ - \phi_-}$$ up to an (irrelevant) integration constant

$$\rightsquigarrow [\vartheta(x), \varphi(y)] = i\,\frac{\operatorname{sgn}(x-y)}{2}, \text{ since } \partial_x[\vartheta(x), \varphi(y)] \underset{\text{Def. } \vartheta}{=}$$

$$= -[\Pi(x), \varphi(y)] = i\delta(x-y). \checkmark$$

↳ $\phi_+ = \frac{1}{2}(\varphi + \vartheta)$, $\phi_- = \frac{1}{2}(\varphi - \vartheta)$.

- Back to expressing fermionic creation operators in terms of Bose fields: $\tilde{\Psi}^+(x') = A e^{i\lambda \int_{-\infty}^{x'} dy\, \Pi(y)} \underset{\text{Def. } \vartheta}{=} A e^{-i\lambda\vartheta(x')}$, where A must commute with ρ.

- To construct fermionic operators for right (left) movers, we can only use bosonic fields that depend on $(x-vt)$ $((x+vt))$. With $2\phi_+ = \rho + \vartheta$, $2\phi_- = \rho - \vartheta$, we arrive at the ansatz: $\psi_+^\dagger(x) = A_+ e^{-2i\lambda\phi_+(x)}$, $\psi_-^\dagger(x) = A_- e^{2i\lambda\phi_-(x)}$.

- The remaining "parameters" A_+, A_-, λ must be chosen such that the fermionic operators obey the algebra

$$\boxed{\begin{aligned} \{\psi_\ell(x), \psi_{\ell'}^\dagger(x')\} &= \delta_{\ell,\ell'}\,\delta(x-x') \\ \{\psi_\ell(x), \psi_{\ell'}(x')\} &= 0\,, \quad \ell,\ell' = \pm \end{aligned}}$$

From $\psi_\ell^\dagger(x)\,\psi_\ell^\dagger(y) \overset{!}{=} -\psi_\ell^\dagger(y)\,\psi_\ell^\dagger(x)$, we can determine λ:

$$\Psi_+^\dagger(x)\Psi_+^\dagger(y) \sim e^{-i\lambda(\varphi(x)+\vartheta(x))} e^{-i\lambda(\varphi(y)+\vartheta(y))} =$$

$$\boxed{e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \text{ if } [A,B] \text{ number}}$$

$$e^{\underbrace{-i\lambda(\varphi(x)+\vartheta(x)+\varphi(y)+\vartheta(y))}_{A+B}} e^{-\frac{\lambda^2}{2}([\varphi(x),\vartheta(y)]+[\vartheta(x),\varphi(y)])} =$$

$$e^{A+B} e^{i\frac{\lambda^2}{2}\operatorname{sgn}(x-y)}$$

$$\Psi_+^\dagger(y)\Psi_+^\dagger(x) = \ldots = e^{A+B} e^{-i\frac{\lambda^2}{2}\operatorname{sgn}(x-y)}$$

$$\rightsquigarrow \Psi_+^\dagger(x)\Psi_+^\dagger(y) = \Psi_+^\dagger(y)\Psi_+(x) e^{-i\lambda^2\operatorname{sgn}(x-y)}$$

$$\rightsquigarrow e^{\pm i\lambda^2} \overset{!}{=} -1 \rightsquigarrow \lambda^2 = \pi$$

$$\rightsquigarrow \boxed{\Psi_+^\dagger(x) = A_+ e^{-i\sqrt{4\pi}\,\phi_+(x)} \; ; \; \Psi_-^\dagger(x) = A_- e^{i\sqrt{4\pi}\,\phi_-(x)}}$$

B.2.b Constructive Bosonization

In the following, we will more carefully derive that (for $L \to \infty$)

$$\psi_\ell(x) = \frac{1}{\sqrt{2\pi a}}\, \eta_\ell\, e^{i\ell\sqrt{4\pi}\,\phi_\ell(x)},$$

where a is a short-distance cutoff, and η_ℓ are the fermionic Klein factors that satisfy $\{\eta_\ell, \eta_{\ell'}^\dagger\} = 2\delta_{\ell\ell'}$ and commute with ρ!

- Starting point: Particle-hole excitations described by Fourier components of the particle density: $\rho_{\ell,q} = \sum_k c_{\ell,k}^\dagger c_{\ell,k+q}$, $\ell = \pm$

  - To avoid/control unphysical effects of extending the linearized spectrum of the Tomonaga-Luttinger Hamiltonian (see B.b.1) beyond the cutoff Λ, we introduce the truncation operator

$$W_{\ell,k} = \begin{cases} 1 & \text{for } \varepsilon_{\ell,k} \in [-\Lambda, \Lambda]. \\ 0 & \text{otherwise.} \end{cases}$$

and the truncated density components

$$\tilde{\rho}_{l,q} = \sum_k W_{l,k} W_{l,k+q} \, c^\dagger_{l,k} c_{l,k+q} \, ,$$

which obey the commutation relations (see Exercise 5.1)

$$\boxed{[\tilde{\rho}_{l,q}, \tilde{\rho}_{l',q'}] = l \, \delta_{l,l'} \delta_{q,-q'} \frac{Lq}{2\pi}}$$

for energies/temperatures much smaller than Λ.
Formally generally valid for $\Lambda \to \infty$.

- Bose operators and their representation in Fock space

$$b_q = \frac{-i}{\sqrt{|\frac{Lq}{2\pi}|}} \sum_{l=\pm} \underbrace{\theta(lq)}_{\text{step-function}} \tilde{\rho}_{l,q} \, , \quad b_q^\dagger = \frac{i}{\sqrt{|\frac{Lq}{2\pi}|}} \sum_{l=\pm} \theta(lq) \, \tilde{\rho}_{l,-q} \, , \text{ where } \underline{q \neq 0}$$

satisfy

$$\boxed{[b_q, b^\dagger_{q'}] = \delta_{q,q'} \, , \quad [b_q, b_{q'}] = 0}$$

- Physical interpretation: For $q>0$ b_q lowers the wave-vector k of all right-movers by q, if $k-q$ is an empty state. b_{-q} acts similarly on left-movers.

  $\rightsquigarrow$ b_q ($b_q^\dagger$) annihilate (create) electron-hole pairs and conserve the number N_ℓ of particles on branch ℓ.

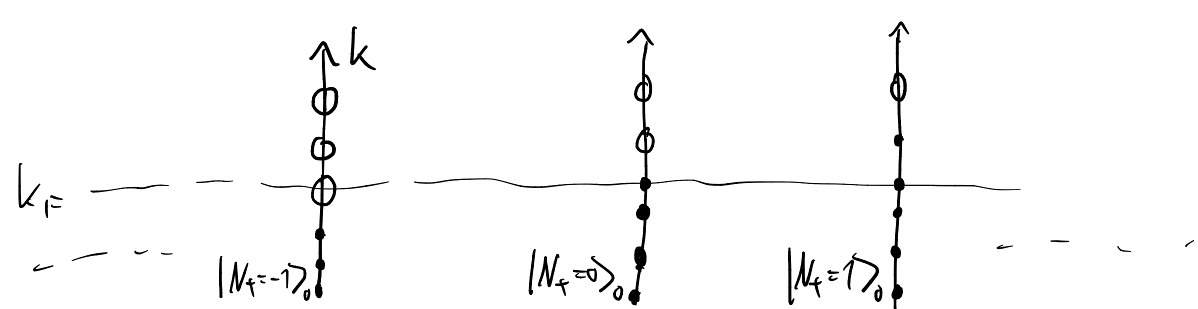
- Fock space for $\ell=+$ branch (analogous for $\ell=-$)

  ↳ Let $|N_+=0\rangle_0$ be the Fermi-sea with all $k\leq k_F$ occupied and all $k>k_F$ empty.

  ↳ In $|N_+=1\rangle_0$ ($|N_+=2\rangle_0$) also the first (first two) momenta above k_F are occupied.

  ↳ $|N_+=-1\rangle_0$ ($|N_+=-2\rangle_0$) results from $|N_+=0\rangle_0$ by removing the energetically highest (two highest) particles from the Fermi sea.

Schematically



↳ A complete basis in Fock space of right-movers is then given by

$$\left| N_+ , \{m_q\}_{q>0} \right\rangle = \prod_{q>0} \frac{(b_q^+)^{m_q}}{\sqrt{m_q!}} \left| N_+ \right\rangle_0 , \quad N_+ = 0, \pm 1, \pm 2, \ldots \quad m_q = 0, 1, 2, \ldots$$

- By analogy for the "$l = -$" branch, the analogous basis reads

$$\left| N_- , \{m_q\}_{q<0} \right\rangle = \prod_{q<0} \frac{(b_q^+)^{m_q}}{\sqrt{m_q!}} \left| N_- \right\rangle_0$$

- Combining the two branches, a complete basis in Fock space is given by

$$\boxed{\left| N_+ , N_- , \{m_q\} \right\rangle = \prod_{q \neq 0} \frac{(b_q^+)^{m_q}}{\sqrt{m_q!}} \left| N_+ , N_- \right\rangle_0 \quad N_l = 0, \pm 1, \pm 2, \ldots , \; l = \pm \quad m_q = 0, 1, 2, \ldots}$$