Analytical Tools for Quantum Many-Body Physics (WS 2022/23)
Exercise Sheet 2

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1. **Self-energy of the Fermi-Hubbard model**
Consider the Fermi-Hubbard model described by the Hamiltonian

$$\hat{H} = \sum_{i,j,\sigma} t_{i,j}\, \hat{c}^\dagger_{i,\sigma}\hat{c}_{j,\sigma} + \frac{U}{2}\sum_{i,\sigma} \hat{n}_{i,\sigma}\hat{n}_{i,-\sigma}\ , \tag{1}$$

where the $t_{i,j}$ denote the (spin-independent) hopping amplitudes, $U > 0$ the Hubbard interaction strength and $\hat{n}_{i,\sigma} = \hat{c}^\dagger_{i,\sigma}\hat{c}_{i,\sigma}$. Calculate the (causal) Green's function $G^{\mathrm{c}}_{\sigma,\sigma'}(k;\omega)$ in the frequency-momentum space to first order in perturbation theory in U.
Then, calculate the self-energy $\Sigma_{\sigma,\sigma'}(k;\omega)$ to first order perturbation theory, and use Dyson equation to resum the self-energy contributions into the Green's function $G^{\mathrm{c}}_{\sigma,\sigma'}(k;\omega)$. Compare this result with the structure of a mean-field decoupling of the interaction term within the Hartree-Fock approximation.
Finally, draw the Feynman diagrams contributing to the self-energy to second order in the interaction. Which ones contribute? Translate the diagrams into mathematical expressions.

2. **Thomas-Fermi screening of Coulomb interaction**
Consider the degenerate electron gas (*jellium model*) in three dimensions, described by the following Hamiltonian:

$$\hat{H} = \sum_{\boldsymbol{k},\sigma} \varepsilon^0_{\boldsymbol{k}}\, \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma} + \frac{1}{2N}\sum_{\boldsymbol{k},\boldsymbol{k}',\sigma,\sigma'}\sum_{\boldsymbol{q}\neq 0} V_{\boldsymbol{q}}\, \hat{c}^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}\hat{c}^\dagger_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}\hat{c}_{\boldsymbol{k}',\sigma'}\hat{c}_{\boldsymbol{k},\sigma} \tag{2}$$

where $\varepsilon^0_{\boldsymbol{k}} = k^2/(2m) - \varepsilon_F$ and $V_{\boldsymbol{q}} = 1/q^2$, with $\hbar$, e (electron charge) and ϵ_0 (vacuum dielectric constant) set to 1. Here, N denotes the total number of sites (volume) and $\varepsilon_F = k_F^2/(2m)$ is the Fermi energy. Calculate the vacuum polarization diagram (*bubble* diagram) in momentum-frequency space to lowest order in perturbation theory. In the thermodynamic limit ($N \to \infty$), you should arrive at an expression of the following form:

$$\Pi(\boldsymbol{k},\omega) = 2\int \frac{d^3q}{(2\pi)^3}\,\Theta(|\boldsymbol{k}+\boldsymbol{q}| - k_F)\,\Theta(k_F - q)\left[\frac{1}{\omega + \varepsilon^0_{\boldsymbol{q}} - \varepsilon^0_{\boldsymbol{k}+\boldsymbol{q}} + i\,0^+} - \frac{1}{\omega - \varepsilon^0_{\boldsymbol{q}} + \varepsilon^0_{\boldsymbol{k}+\boldsymbol{q}} - i\,0^+}\right]. \tag{3}$$

At this point, you may use the result by Lindhard, which in the static case ($\omega = 0$) reads as

$$\Pi(\boldsymbol{k},0) = -\frac{2m\,k_F}{(2\pi)^2}\left[1 + \frac{k_F^2 - (k/2)^2}{k_F k}\,\log\left|\frac{k_F + k/2}{k_F - k/2}\right|\right]. \tag{4}$$

Derive an expression for the static effective interaction $V^{\mathrm{eff}}(\boldsymbol{k},0)$ in the long-wavelength limit, using the Dyson equation to resum the above lowest-order contributions. This resummation of bubble diagrams corresponds to the random phase approximation (RPA). Fourier-transform $V^{\mathrm{eff}}(\boldsymbol{k},0)$ to derive the effective Yukawa interaction potential in real space:

$$V^{\mathrm{eff}}(\boldsymbol{r}) = \frac{\mathrm{e}^{-k_{\mathrm{TF}}\,r}}{4\pi\,r}, \tag{5}$$

where $k_{\mathrm{TF}}^2 = 2m\,k_F/(2\pi)^2$ is the Thomas-Fermi *screening* wavevector. At distances $r \gg k_{\mathrm{TF}}^{-1}$, the presence of the other charged particles leads to a screening of the initially long-ranged Coulomb repulsion.

Bonus: Prove Eq. (4). You may find the following identity useful:

$$\int_0^a dx\, x\,\log\left|\frac{x+b}{b-x}\right| = a\,b + \frac{1}{2}(a^2 - b^2)\log\left|\frac{a+b}{a-b}\right|\ .$$