Analytical Tools for Quantum Many-Body Physics (WS 2022/23)
Exercise Sheet 3

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1. **Debye-Hückel screening of the Coulomb potential at high temperature**
The objective of this problem is to derive the screening length of the Coulomb potential in the non-degenerate limit of the electron gas. To this end, consider the Hamiltonian of the interacting electron gas (*jellium model*) in three dimensions:

$$\hat{H} - \mu\,\hat{N} = \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k}}\,\hat{c}^{\dagger}_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma} + \frac{1}{2V}\sum_{\boldsymbol{k},\boldsymbol{k}',\sigma,\sigma'}\sum_{\boldsymbol{q}\neq 0} V_{\boldsymbol{q}}\,\hat{c}^{\dagger}_{\boldsymbol{k}+\boldsymbol{q},\sigma}\hat{c}^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q},\sigma'}\hat{c}_{\boldsymbol{k}',\sigma'}\hat{c}_{\boldsymbol{k},\sigma} \tag{1}$$

where $\varepsilon_{\boldsymbol{k}} = k^2/(2m) - \mu$, μ is the chemical potential, $V_{\boldsymbol{q}} = 1/q^2$ ($\hbar$, the electron charge e and the vacuum dielectric constant ϵ_0 are set to 1) and with V denoting the total number of sites (volume).

(a) Derive the expression for the vacuum polarization diagram, i.e. the lowest order polarization insertion (*bubble* diagram, cf. Problem 2 on Sheet 2) at finite temperature using the Matsubara formalism in momentum-frequency space. You should arrive at an expression of the following form:

$$\mathit{\Pi}(\boldsymbol{k},\nu_n) = 2\int \frac{d^3q}{(2\pi)^3}\,\frac{n^{(0)}_{\boldsymbol{q}} - n^{(0)}_{\boldsymbol{k}+\boldsymbol{q}}}{i\,\nu_n - (\varepsilon_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon_{\boldsymbol{q}})}\ , \tag{2}$$

where $n^{(0)}_{\boldsymbol{q}} = (e^{\beta\varepsilon_{\boldsymbol{q}}} + 1)^{-1}$.

(b) Evaluate the static ($\nu_n = 0$) vacuum polarization in the non-degenerate (high temperature, low density) case. The result, expressed in terms of the thermal wavelength $\lambda^2_{\mathrm{th}} = 2\pi\beta/m$, should read as

$$\mathit{\Pi}(\boldsymbol{k},0) = -2\beta\,e^{\beta\mu}\lambda_{\mathrm{th}}^{-3}\,\varphi(\lambda_{\mathrm{th}}q)\ , \tag{3}$$

where

$$\varphi(y) = \frac{1}{\pi\,y}\int_0^{\infty} dx\,x\,e^{-\frac{x^2}{4\pi}}\log\left|\frac{2x+y}{2x-y}\right|\ .$$

(c) Derive an expression for the static effective interaction $V^{\mathrm{eff}}(\boldsymbol{k},0)$ in the long-wavelength limit, using the Dyson equation to resum the above lowest-order contributions. By Fourier-transforming $V^{\mathrm{eff}}(\boldsymbol{k},0)$ derive the effective interaction potential in real space, showing that it is of the Yukawa type having the form:

$$V^{\mathrm{eff}}(\boldsymbol{r}) = \frac{\mathrm{e}^{-k_{\mathrm{DH}}\,r}}{4\pi\,r} \tag{4}$$

with the Debye-Hückel *screening* wavevector $k^2_{\mathrm{DH}} = 4\beta\,n^{(0)}$, where $n^{(0)} = 2e^{\beta\mu}\lambda_{\mathrm{th}}^{-3}$.

Hint: In the non-degenerate regime, i.e. high temperature and low density, respectively, the Fermi-Dirac distribution is well approximated by the Maxwell-Boltzmann one, i.e. $n^{(0)}_{\boldsymbol{k}} = e^{-\beta\varepsilon_{\boldsymbol{k}}}$.