Analytical Tools for Quantum Many-Body Physics (WS 22/23)
Exercise Sheet 4

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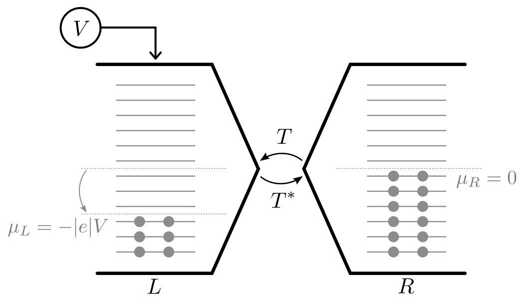
The goal of this exercise sheet is to practically apply the Keldysh formalism to an electron transport problem, namely the tunneling through a barrier between two conducting leads modeled within the *tunneling Hamiltonian formalism.* A schematic representation of the setup is shown in the figure below.



1. **Electron tunneling between two leads: resummation of the perturbative series**
Consider the scenario represented in the above figure, where two conducting leads, left L and right R, are weakly coupled by electron tunnelling. The two leads are described by Hamiltonians $\hat{H}_L$ and $\hat{H}_R$ and kept at *different* chemical potentials μ_L and μ_R with $\Delta\mu = \mu_R - \mu_L = |e|V$. This corresponds to having the left lead L *gated* by voltage V shifting its chemical potential, thereby creating a non-equilibrium situation where electrons flow from one lead to the other. The tunneling is described by the Hamiltonian $\hat{H}_T$. The total Hamiltonian $\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_T$ is thus specified by

$$\hat{H}_L = \sum_{\boldsymbol{k},\sigma} \varepsilon^L_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{c}_{\boldsymbol{k},\sigma} \quad , \quad \hat{H}_R = \sum_{\boldsymbol{q},\sigma} \varepsilon^R_{\boldsymbol{q}} \hat{d}^\dagger_{\boldsymbol{q},\sigma}\hat{d}_{\boldsymbol{q},\sigma} \quad ,$$
$$\hat{H}_T = \sum_{\boldsymbol{k},\boldsymbol{q},\sigma} \left(T\hat{c}^\dagger_{\boldsymbol{k},\sigma}\hat{d}_{\boldsymbol{q},\sigma} + T^*\hat{d}^\dagger_{\boldsymbol{q},\sigma}\hat{c}_{\boldsymbol{k},\sigma}\right) \quad , \tag{1}$$

where the $\hat{c}$ ($\hat{d}$) operators annihilate electrons in lead L (R), the modulus of the electron charge is $|e|$, and T is a weak tunneling amplitude.

(a) Show that the current between the leads

$$I(V,t) = -|e| \left\langle \frac{d}{dt}\hat{N}_R(t)\right\rangle \quad , \tag{2}$$

where $\hat{N}_R(t) = \sum_{\boldsymbol{q},\sigma} \hat{d}^\dagger_{\boldsymbol{q},\sigma}(t)\hat{d}_{\boldsymbol{q},\sigma}(t)$ is the number operator in the lead R in the Heisenberg picture, can be written as:

$$I(V,t) = \frac{2|e|}{\hbar}\,\mathrm{Re}\left(T^* \sum_{\boldsymbol{k},\boldsymbol{q},\sigma} F^{-+}_{\boldsymbol{k},\boldsymbol{q},\sigma}(0)\right) \quad , \tag{3}$$

where $F^{-+}_{\boldsymbol{k},\boldsymbol{q},\sigma}(t) = i\,\langle \hat{d}^\dagger_{\boldsymbol{q},\sigma}(0)\hat{c}_{\boldsymbol{k},\sigma}(t)\rangle$ is the "$-+$" element of the (real-time formalism) matrix Green's function $\hat{F}_{\boldsymbol{k},\boldsymbol{q},\sigma}(t)$ mixing states in left and right leads.

(b) Treating $\hat{H}_T$ as the perturbation, compute the current by explicitly resumming the perturbation series in the tunneling amplitude T. Assume zero temperature leads with approximately constant density of states around the characteristic frequencies of the problem. The final expression for the current should read as

$$I(V,t) = \frac{4|e|}{\hbar}\frac{2\pi^2|T|^2}{\hbar}\int_0^{\frac{|e|V}{\hbar}} \frac{d\omega}{2\pi}\,\frac{\rho_L(\omega-|e|V/\hbar)\,\rho_R(\omega)}{\left[1+\frac{\pi^2|T|^2}{\hbar^2}\rho_L(\omega-|e|V/\hbar)\,\rho_R(\omega)\right]^2} \ , \tag{4}$$

where $\rho_{L/R}(\omega) \equiv \sum_{\boldsymbol{k}} \delta(\omega - \varepsilon_{\boldsymbol{k}}^{L/R})$ is the density of states in the left/right lead.

(c) In the limit of small V show that the conductance is then approximated by

$$G \approx G_0 \frac{T_{\text{eff}}^2}{\left(1+T_{\text{eff}}^2/4\right)^2} \ , \tag{5}$$

where $G_0 = \frac{2e^2}{h}$ denotes the quantum of conductance, and $T_{\text{eff}}^2 = \left(\frac{2\pi|T|}{\hbar}\rho_L(0)\right)\left(\frac{2\pi|T|}{\hbar}\rho_R(0)\right)$.

Hints: You may find the calculation easier in frequency space and when using the Larkin-Ovchinnikov (or *triagonal*) representation of the matrix Green's function, which reads as:

$$\bar{G}(1,2) = \begin{pmatrix} G^R(1,2) & G^K(1,2) \\ 0 & G^A(1,2) \end{pmatrix} \ .$$

Assuming the density of states of the leads to be approximately constant, you can use

$$g_{L/R}^{R/A}(\omega) \equiv \sum_{\boldsymbol{k}} G_{0,L/R}^{R/A}(\boldsymbol{k},\omega) \approx \mp i\pi \sum_{\boldsymbol{k}} \delta\big(\omega - (\varepsilon_{\boldsymbol{k}}^{L/R} - \mu_{L/R})\big) = \mp i\pi\, \rho_{L/R}(\omega+\mu_{L/R}) \ ,$$
$$g_{L/R}^{K}(\omega) \equiv \sum_{\boldsymbol{k}} G_{0,L/R}^{K}(\boldsymbol{k},\omega) \approx -2\pi i[1-2n_F(\omega+\mu_{L/R})]\rho_{L/R}(\omega+\mu_{L/R})$$

in the integrals over frequency, where $G_{0,L/R}^{R/A}(\boldsymbol{k},\omega)$ are the retarded/advanced unperturbed Green's functions of the left/right lead, and $G_{0,L/R}^{K}(\boldsymbol{k},\omega)$ are the corresponding Keldysh Green's functions.