Analytical Tools for Quantum Many-Body Physics (WS 2022/23)
Exercise Sheet 5

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Online: Thu, 05 Jan 2023
Discussion: Thu, 19 Jan 2023

The goal of this exercise sheet is to prove some important relations for the framework of bosonization.

1. **Commutation relations of the Fourier components of the density operator**
   Consider the density of fermions with chirality $r = L/R = \pm$ defined as $\rho_r(x) = c_r^\dagger(x)c_r(x)$, where the $c_r^\dagger(x)$ and $c_r(x)$ are the creation and annihilation operators of r-fermions at position x. The Fourier components of $\rho_r(x)$ are defined by $\rho_r(x) = \frac{1}{L}\sum_k e^{ikx}\rho_{k,r}$, where L is the length of the system and

   $$\rho_{k,r} = \sum_q c_{q-k,r}^\dagger c_{q,r} \ , \tag{1}$$

   with the creation and annihilation operators $c_{q,r}^\dagger$ and $c_{q,r}$ of r-fermions with momentum q. Show that

   $$\left[\rho_{k,r}\ ,\rho_{k',r'}^\dagger\right] = \delta_{r,r'}\delta_{k,k'}\,\frac{rkL}{2\pi} \ , \tag{2}$$

   which is reminiscent of a bosonic commutation relation, thus hinting to the fact that bosonic operators $b_k^\dagger$ and b_k can be constructed from $\rho_{k,r}^\dagger$ and $\rho_{k,r}$, respectively.

   Hints: Note that since the Hilbert space of the theory is infinite-dimensional (the spectrum is unbounded from below), the operators contain divergences. Use normal ordering with respect to the Fermi sea ground-state to separate the (divergent) ground-state expectation value of the operators from their (finite) normal ordered contributions.

2. **Crucial identity for expectation values in bosonic systems**
   Consider a bosonic system with Hamiltonian $H = \sum_j \varepsilon_j\left(b_j^\dagger b_j + \frac{1}{2}\right)$. Show that for an operator B linear in the bosonic operators $b_j^\dagger$ and b_j, i.e. $B = \sum_j\left(\alpha_j b_j + \gamma_j b_j^\dagger\right)$, the thermal expectation value of the exponential operator e^B is given by

   $$\left\langle e^B\right\rangle = e^{\frac{1}{2}\langle B^2\rangle} \ , \tag{3}$$

   where $\langle O\rangle = \frac{1}{Z}\,\mathrm{tr}\left(e^{-\beta H}O\right)$ denotes the thermal expectation value of an operator O.