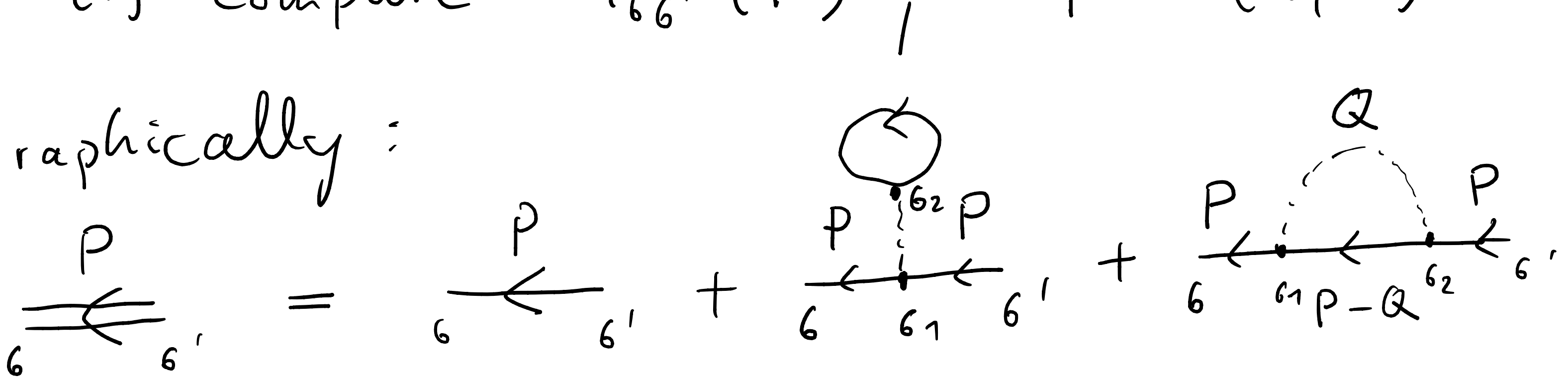
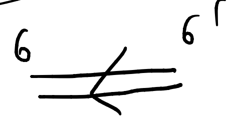
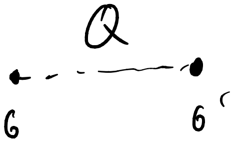
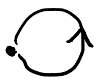
1. (i) Compute $G^c_{\sigma\sigma'}(P)$, $P = (k, \omega)$

Graphically:

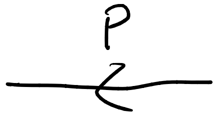


Decoding table:

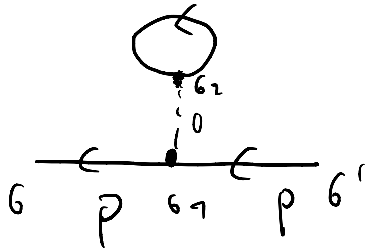
	$i\,G^c_{\sigma\sigma'}(P)$
	$i\,G^{c,0}_{\sigma\sigma'}(P)\,\delta_{\sigma,\sigma'}$
	$-i\,U(Q)\,\delta_{\sigma,-\sigma'} = -i\frac{U}{2}\delta_{\sigma,-\sigma'}$ $\uparrow$ Hubbard model
	$n_0(\mu)$

For t_{ij}, assume NN-hopping on a cubic lattice with unit lattice constant → $\varepsilon_k = -2t(\cos k_x + \cos k_y + \cos k_z) - \mu$, $H_0 = -t\sum_{\langle i,j\rangle,\sigma} c^\dagger_{i,\sigma} c_{j,\sigma} - \mu\hat{N}$

Diagram by diagram:

(a) 

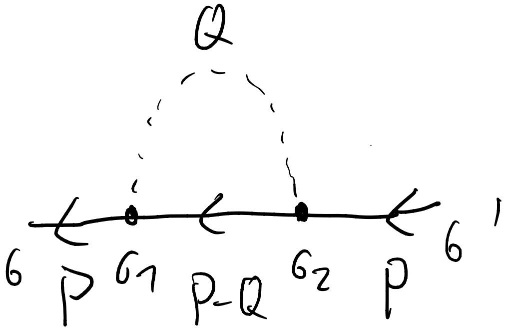
$$= i G^{c,0}_{\sigma,\sigma'}(k,\omega) = i\,\delta_{\sigma,\sigma'}\,\frac{1}{\omega-\varepsilon_k+i0^+\mathrm{sgn}(\varepsilon_k)} .$$

(b) 

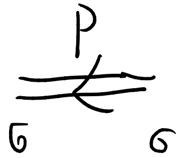
$$= \sum_{\sigma_1,\sigma_2}\left(-i\frac{U}{2}\right)\delta_{\sigma_1,-\sigma_2}\; iG^{c,0}_{\sigma,\sigma_1}(P)\; iG^{c,0}_{\sigma_1,\sigma'}(P)\; n_0(\mu)$$

$$= \sum_{\sigma_1}\left(-i\frac{U}{2}\right) iG^{c,0}_{\sigma,\sigma_1}(P)\; iG^{c,0}_{\sigma_1\sigma'}(P)\; n_0(\mu) =$$

$$= \frac{i\,\delta_{\sigma,\sigma'}\,U\,n_0(\mu)}{2}\left(\frac{1}{\omega-\varepsilon_k+i0^+\mathrm{sign}(\varepsilon_k)}\right)^2$$

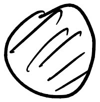
(c) 

$$= \sim \delta_{\sigma_1,\sigma_2}\,\delta_{\sigma_1,-\sigma_2} = 0$$

$\rightsquigarrow$ 

$$= i\delta_{\sigma,\sigma'}\left[\frac{1}{\omega-\epsilon_k+i0^+\mathrm{sgn}(\epsilon_k)} + \frac{U n_0(\mu)}{2}\left(\frac{1}{\omega-\epsilon_k+i0^+\mathrm{sgn}(\epsilon_k)}\right)^2\right] + \mathcal{O}(U^2)$$

(ii) Self-energy

$-i\Sigma(p)_{\sigma\sigma'}$ =  = 

$+\ \mathcal{O}(U^2) =$

$$= -i\delta_{\sigma\sigma'}\frac{U n_0(\mu)}{2}$$

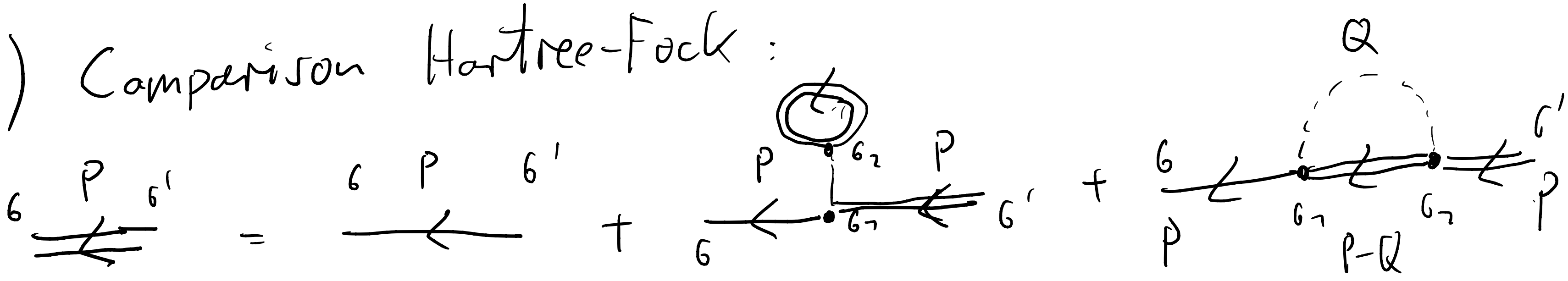
(iii) Resummation in Dyson equation

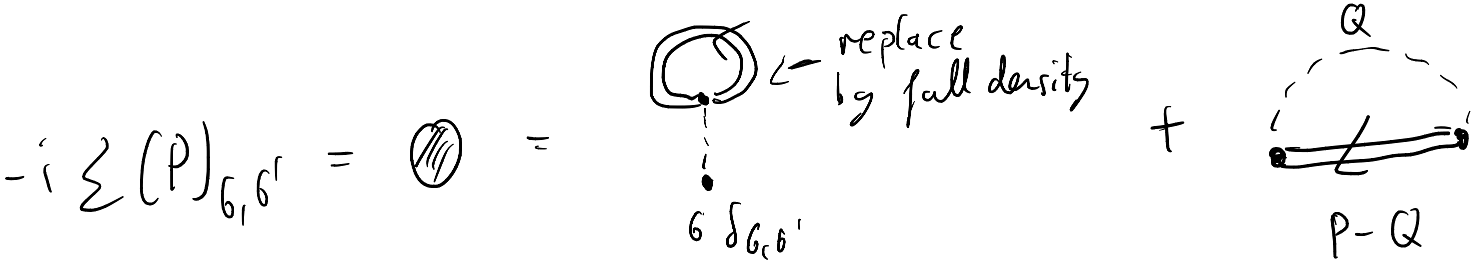
$$G^c(k,\omega,\sigma) = \left(G^{c,0}(k,\omega,\sigma)^{-1} - \Sigma(k,\omega,\sigma)\right)^{-1} = \frac{1}{\omega-\epsilon_k+i0^+\mathrm{sgn}(\epsilon_k)-\frac{Un_0(\mu)}{2}}$$

↑ diagonal in spin

↑ $=\Sigma(k,\omega,\sigma)$

(iv) Comparison Hartree-Fock:





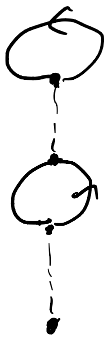
Translation-invariance and diagonal spin-structure are then externally imposed assumptions that may be violated by the true solution (magnetism and/or spatial order). This can be accomodated in a Hartree-Fock calculation by considering a larger unit cell or abandoning the assumption of translation invariance all together.

(V) Self-energy diagrams to second order

=

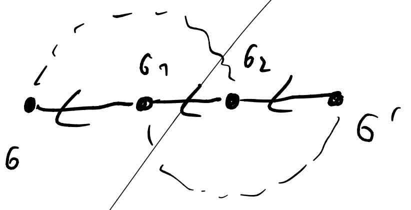
$\sim \delta_{\sigma_1,\sigma_2}\delta_{\sigma_2,\sigma_3}\delta_{\sigma_2,-\sigma_3} = 0$

+



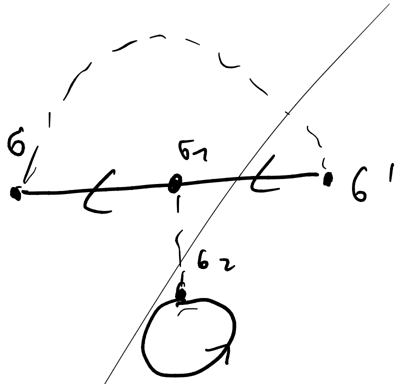
✓

+



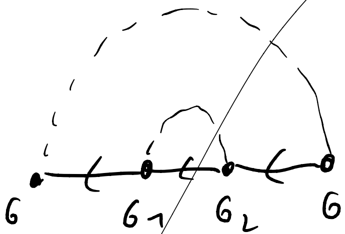
$\sim \delta_{\sigma,\sigma_1}\delta_{\sigma_1,\sigma_2}\delta_{\sigma,-\sigma_2} = 0$

+



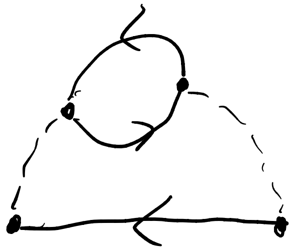
$\sim \delta_{\sigma,\sigma_1}\delta_{\sigma_1,\sigma'}\delta_{\sigma,-\sigma'} = 0$

+



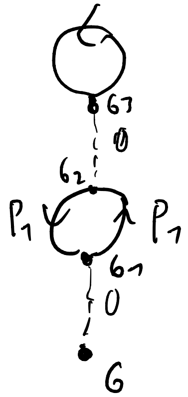
$\sim \delta_{\sigma_1,\sigma_2}\delta_{\sigma_1,-\sigma_2} = 0$

+



✓

Translate into expression:



$\delta_{\sigma,\sigma'}$

$$= \delta_{\sigma,\sigma'} \sum_{\sigma_1,\sigma_2,\sigma_3} \left(-\frac{i}{2} U\right) \delta_{\sigma,-\sigma_1} \left(-\frac{i}{2} U\right) \delta_{\sigma_1,-\sigma_2} \int \frac{d\omega_1}{2\pi} \int \frac{d^3k_1}{(2\pi)^3} \times$$

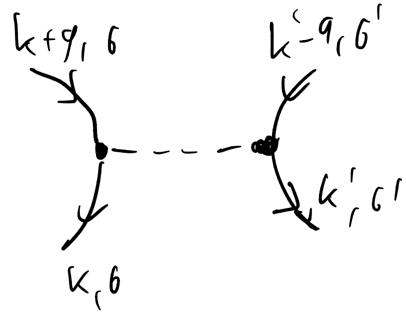
$$\times (-1) \; i G^{c,0}_{\sigma_1,\sigma_2}(k_1,\omega_1) \, i G^{c,0}_{\sigma_2,\sigma_1}(k_1,\omega_1) \, n_0(\mu) .$$

Analogous for polarization diagram.

2.) (i) 3D Jellium model in reciprocal space:

$$H = \sum_{k,\sigma} \varepsilon_k c^{\dagger}_{k,\sigma} c_{k,\sigma} + \frac{1}{2N} \sum_{k,k',\sigma,\sigma'} \sum_{q \neq 0} V_q \, c^{\dagger}_{k+q,\sigma} c^{\dagger}_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma}$$

↑ $q = 0$ cancelled by Jellium background

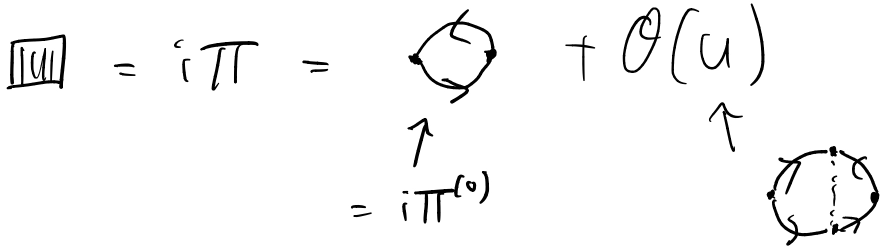


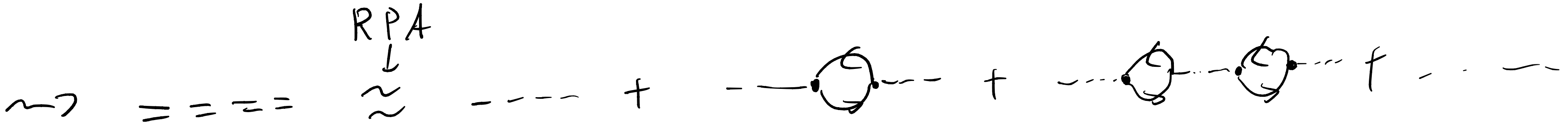
(spin conserved at every vertex)

$$\varepsilon_k = \frac{k^2}{2m} - \varepsilon_F \,, \quad \varepsilon_F = \frac{k_F^2}{2m} \,, \quad V_q = \frac{1}{q^2}$$ (F.T. of Coulomb vertice)

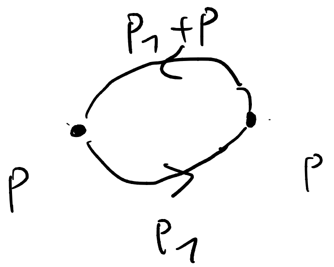
(ii) Dyson equation for polarization operator

with





$$i\Pi^{(0)}(\underbrace{k,\omega}_{p}) =$$



$$= 2\int\frac{d\omega_1}{2\pi}\int\frac{d^3k_1}{(2\pi)^3}(-1)\, i\, G_c^0(k_1+k,\omega+\omega_1)\, i\, G_c^0(k_1,\omega_1)$$

spin summation

Fermi-loop

$$\frac{1}{N}\sum_{k_1} \longrightarrow \int\frac{d^3k_1}{(2\pi)^3}$$

outside FS

inside FS ($\varepsilon_k < 0$)

$$\boxed{G_c^0(k,\omega) = \frac{\theta(k-k_F)}{\omega-\varepsilon_k+i0^+} + \frac{\theta(k_F-k)}{\omega-\varepsilon_k-i0^+}}$$

Fermi-sphere (FS)

$$\rightsquigarrow G^0_c(k+k_1, \omega+\omega_1)\, G^0_c(k_1, \omega_1) = \frac{\theta(|k_1+k|-k_F)\,\theta(|k_1|-k_F)}{(\omega+\omega_1-\varepsilon_{k+k_1}+i0^+)(\omega_1-\varepsilon_{k_1}+i0^+)} \quad \swarrow \text{both outside FS} \qquad (\mathrm{I})$$

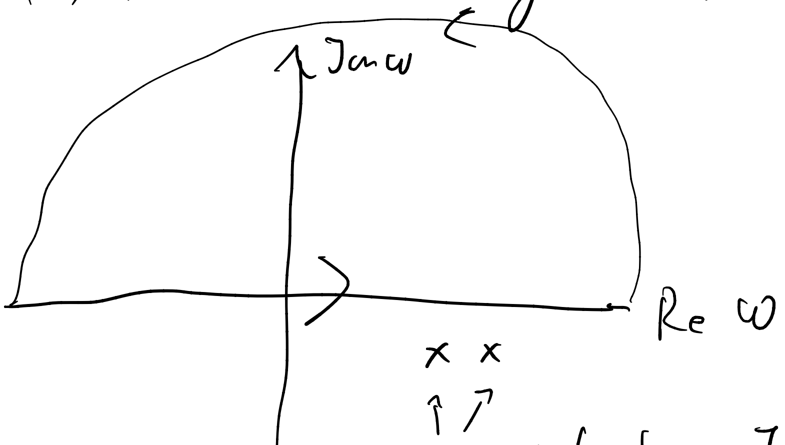
$$+ \frac{\theta(|k_1+k|-k_F)\,\theta(k_F-|k_1|)}{(\cdots\cdots +i0^+)(\cdots\cdots -i0^+)} \quad (\mathrm{II}) \quad \leftarrow \text{one in one out}$$

$$+ \frac{\theta(k_F-|k_1+k|)\,\theta(|k_1|-k_F)}{(\cdots\cdots -i0^+)(\cdots\cdots +i0^+)} \quad (\mathrm{III}) \quad \leftarrow \text{one in one out}$$

$$+ \frac{\theta(k_F-|k_1+k|)\,\theta(k_F-|k_1|)}{(\cdots\cdots -i0^+)(\cdots\cdots -i0^+)} \quad \leftarrow \text{both inside FS}$$

$\int \frac{d\omega_1}{2\pi}$ (I)··(IV) solved by complex integration using residue calculu

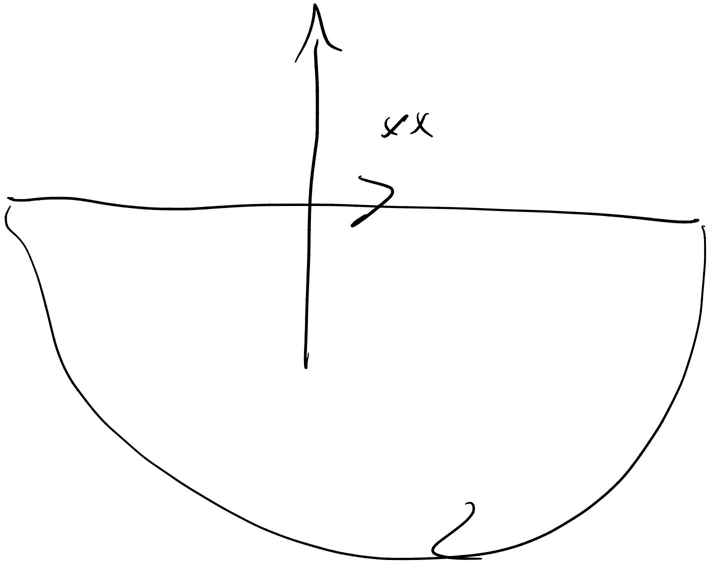
(I)



both poles have Im ω < 0

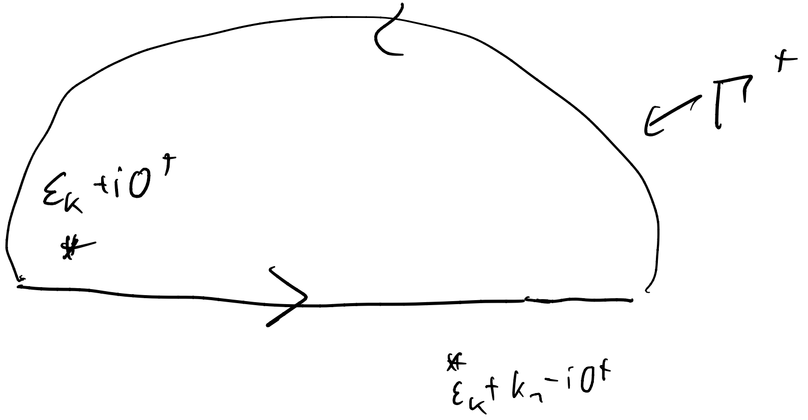
$\rightsquigarrow \int d\omega_1 \, (\text{I}) = 0$

(IV)



$\rightsquigarrow \int d\omega_1 \, (\text{IV}) = 0$

(II)



$$\oint_{\Gamma^+} d\omega_1 \, II = 2\pi i \, \mathrm{Res}\left(II, \omega_1 = \varepsilon_{k_1} + i0^+\right)$$

$$= 2\pi i \lim_{\omega_1 \to \varepsilon_k + i0^+} \left(\omega_1 - \varepsilon_{k_1} - i0^+\right) \cdot (II)$$

$$= 2\pi i \frac{\theta(|k_1 + k| - k_F)\,\theta(k_F - k_1)}{\omega + \varepsilon_{k_1} - \varepsilon_{k_1 + k} + i0^+}$$

Analogously for (III)

For $\int (III)\, d^3k_1$, use $k_1 \to -k_1$, and then $k_1 \to k_1 + k$

$$\rightsquigarrow i\Pi^{(0)}(k,\omega) = 2i \int \frac{d^3k_1}{(2\pi)^3} \theta(|k_1 + k| - k_F)\,\theta(k_F - |k_1|) \left[\frac{1}{\omega + \varepsilon_{k_1} - \varepsilon_{k_1+k} + i0^+} - \frac{1}{\omega + \varepsilon_{k_1+k} - \varepsilon_{k_1} - i0^+}\right]$$

$$\boxed{\log\left|1+\frac{k}{k_F-\frac{k}{2}}\right| = \frac{k}{k_F} + \frac{1}{12}\left(\frac{k}{k_F}\right)^3 + \mathcal{O}(k^5)}$$

⇝ Lindhard :

$$\Pi^{(0)}(k,0) = -\frac{2mk_F}{(2\pi)^2}\left[1+\frac{k_F^{\,2}-\left(\frac{k}{2}\right)^2}{k_F\,k}\overbrace{\log\left|\frac{k_F+\frac{k}{2}}{k_F-\frac{k}{2}}\right|}^{=}\right]$$

$$= -\frac{2mk_F}{(2\pi)^2}\left[1+\left(1-\frac{k^2}{4k_F^{\,2}}+\dots\right)\left(1+\frac{1}{2}\left(\frac{k}{k_F}\right)^2+\dots\right)\right] =$$

$$= -\frac{4mk_F}{(2\pi)^2} + \mathcal{O}(k^2)$$

$$V_{eff}(k,\omega=0) = V(k)\sum_{\ell=0}^{\infty}\left(\Pi^{(0)}(k,0)V(k)\right)^{\ell} = \frac{V(k)}{1-\Pi^{(0)}(k,0)V(k)} =$$

$$= \frac{1}{k^2+\Pi^{(0)}(k,0)} = \frac{1}{k^2+q_{TF}^{\,2}}$$ with $q_{TF} = \frac{4mk_F}{2\pi}$

FT to real space:

$$V_{eff}(r) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{r}}\, \frac{1}{k^2+q_{TF}^2} \overset{\substack{\text{azimuthal}\\ \text{trivial}}}{=} \frac{1}{(2\pi)^2} \int_{-1}^{+1} \overset{u=\cos\vartheta}{du} \int_0^\infty dk\, e^{ikru}\, \frac{k^2}{k^2+q_{TF}^2} =$$

$$\xrightarrow{\substack{\text{perform}\\ \text{u-integral}}} = \frac{1}{(2\pi)^2}\, \frac{2}{r} \int_0^\infty dk\, \frac{k\sin(kr)}{k^2+q_{TF}^2} \underset{\substack{\text{integrand}\\ \text{even in k}}}{=} \frac{1}{(2\pi)^2}\, \frac{1}{r} \int_{-\infty}^{+\infty} dk\, \frac{k\sin(kr)}{k^2+q_{TF}^2} =$$

$$= \frac{1}{(2\pi)^2}\, \frac{1}{2ri} \left[\underbrace{\int_{-\infty}^{+\infty} dk\, \frac{k e^{ikr}}{k^2+q_{TF}^2}}_{①} - \underbrace{\int_{-\infty}^{+\infty} dk\, \frac{k e^{-ikr}}{k^2+q_{TF}^2}}_{②} \right]$$

Use pole structure of $k_2^2 + q_{TF}^2 = (k+iq_{TF})(k-iq_{TF})$ to obtain

$$(1) = 2\pi i \,\mathrm{Res}\left(\frac{k e^{ikr}}{k^2+q_{TF}}, iq_F\right) = i\pi e^{-q_{TF} r}$$

$$(2) = 2\pi i \,\mathrm{Res}\left(\frac{k e^{-ikr}}{k^2+q_{TF}^2}, -iq_{TF}\right) = -i\pi e^{-q_{TF} r}$$

$$\rightsquigarrow \quad \frac{1}{(2\pi)^2}\,\frac{1}{2ri}\left[\textcircled{1} - \textcircled{2}\right] = \boxed{V_{eff}(r) = \frac{1}{4\pi r}\, e^{-q_{TF}\, r}} \, .$$