

1. (i) Compute $G_{66'}^c(P)$, $P = (k, \omega)$

Graphically:

$$\begin{array}{c} P \\ \leftarrow \leftarrow \\ 6 \quad 6' \end{array} = \begin{array}{c} P \\ \leftarrow \\ 6 \quad 6' \end{array} + \begin{array}{c} \text{loop} \\ \leftarrow \leftarrow \leftarrow \\ 6 \quad 6_1 \quad 6' \end{array} + \begin{array}{c} Q \\ \leftarrow \leftarrow \leftarrow \\ 6 \quad 6_1 \quad P-Q \quad 6_2 \quad 6' \end{array}$$

Decoding table:

$\begin{array}{c} 6 \quad 6' \\ \leftarrow \leftarrow \end{array}$	$i G_{66'}^c(P)$
$\begin{array}{c} 6 \quad 6' \\ \leftarrow \end{array}$	$i G_{66'}^{c,0}(P) \delta_{6,6'}$
$\begin{array}{c} Q \\ \cdots \cdots \\ 6 \quad 6' \end{array}$	$-i U(Q) \delta_{6,6'} = -i \frac{U}{2} \delta_{6,6'}$ ↑ Hubbard model
$\begin{array}{c} \text{loop} \end{array}$	$N_0(\mu)$

For t_{ij} , assume NN -hopping on a cubic lattice with unit lattice constant $\rightarrow \epsilon_k = -2t(\cos k_x + \cos k_y + \cos k_z) - \mu$, $H_0 = -t \sum_{\langle i,j \rangle, \sigma} c_{i,\sigma}^+ c_{j,\sigma} - \mu \hat{N}$

Diagram by diagram:

Diagram by diagram:

(a) $\overleftarrow{P} = i G_{b, b'}^{c, 0}(k, \omega) = i \int_{b, b'} \frac{1}{\omega - \epsilon_k + i0^+ \text{sgn}(\epsilon_k)} \cdot$

$$\begin{aligned}
 (b) \quad & \text{Diagram: A horizontal line with points } G, p, b_1, p, b', \text{ and } G'. \text{ A vertical dashed line connects } b_1 \text{ to } b_2 \text{ above it. A circle is drawn around } b_2 \text{ with a dot at } 0. \\
 & = \sum_{b_1, b_2} \left(-i \frac{U}{2} \right) \delta_{b_1, -b_2} i G_{b_1, b_1}^{c_1 0}(p) i G_{b_1, b_1}^{c_1 0}(p) n_0(\mu) \\
 & = \sum_{b_1} \left(-i \frac{U}{2} \right) i G_{b_1, b_1}^{c_1 0}(p) i G_{b_1, b_1}^{c_1 0}(p) n_0(\mu) = \\
 & = \frac{i \sum_{b_1, b_1} U n_0(\mu)}{2} \left(\frac{1}{\omega - \epsilon_k + i 0^+ \text{sign}(\epsilon_k)} \right)^2
 \end{aligned}$$

(c)

$= \sim \int_{g_1, g_2} \int_{g_1, -g_2} = 0$

$$\rightarrow \underline{\underline{P}}_{\vec{6}\vec{6}'} = i\delta_{\vec{6}\vec{6}'} \left[\frac{1}{\omega - \epsilon_k + i0^+ \text{sign}(\epsilon_k)} + \frac{u n_0(\mu)}{2} \left(\frac{1}{\omega - \epsilon_k + i0^+ \text{sign}(\epsilon_k)} \right)^2 \right] + \mathcal{O}(u^2)$$

(ii) Self-energy

$$-i\Sigma(P)_{\vec{6}\vec{6}'} = \text{diagram} = \text{diagram} + \mathcal{O}(u^2) =$$

$$= -i\delta_{\vec{6}\vec{6}'} \frac{u n_0(\mu)}{2}$$

(iii) Resummation in Dyson equation

$$G^c(k, \omega, \vec{6}) = \left(G^{c,0}(k, \omega, \vec{6}) - \Sigma(k, \omega, \vec{6}) \right)^{-1} = \frac{1}{\omega - \epsilon_k + i0^+ \text{sign}(\epsilon_k) - \frac{u n_0(\mu)}{2}}$$

$\uparrow$
 diagonal
 in spin

$\uparrow$
 $= \Sigma(k, \omega, \vec{6})$

(iv) Comparison Hartree-Fock:

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \\
 -i \sum (P)_{G, G'} &= \text{Diagram 5} = \text{Diagram 6} + \text{Diagram 7}
 \end{aligned}$$

Diagram 5: A circle with a dot in the center, labeled $G \delta_{G, G'}$. An arrow points to it from the text "replace by full density".

Diagram 6: A diagram with a dashed arc labeled Q above it. Inside the arc, there are two vertices labeled G_1 and G_2 . Lines connect G_1 to G_2 and G_1 to G_2 (forming a loop). External lines are labeled G , P , G' , and P .

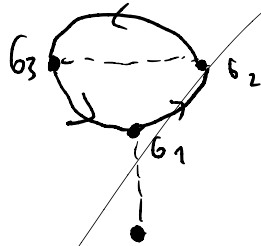
Diagram 7: A diagram with a dashed arc labeled Q above it. Inside the arc, there are two vertices labeled G_1 and G_2 . Lines connect G_1 to G_2 and G_1 to G_2 (forming a loop). External lines are labeled G , P , G' , and P .

Translation-invariance and diagonal spin-structure are then externally imposed assumptions that may be violated by the true solution (magnetism and/or spatial order). This can be accommodated in a Hartree-Fock calculation by considering a larger unit cell or abandoning the assumption of translation invariance all together.

(V) Self-energy diagrams to second order



=



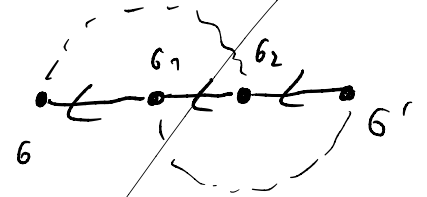
$$\sim \delta_{b_1, b_2} \delta_{b_2, b_3} \delta_{b_2, -b_3} = 0$$

+



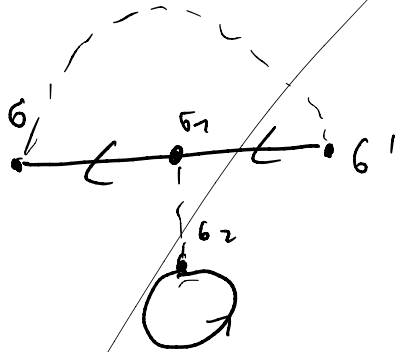
✓

+



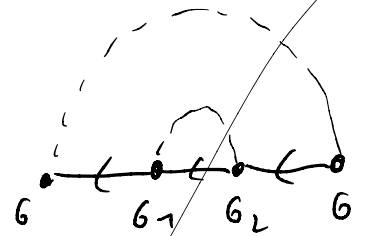
$$\sim \delta_{b, b_1} \delta_{b_1, b_2} \delta_{b_1, -b_2} = 0$$

+



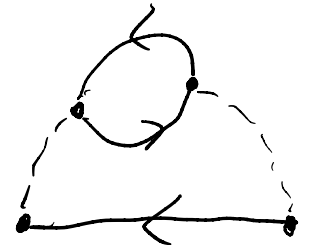
$$\sim \delta_{b, b_1} \delta_{b_1, b_2} \delta_{b_1, -b_2} = 0$$

+



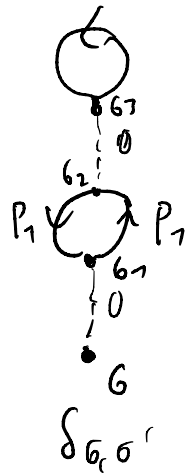
$$\sim \delta_{b_1, b_2} \delta_{b_1, -b_2} = 0$$

+



✓

Translate into expression:



$$= \delta_{b,b'} \sum_{b_1, b_2, b_3} \left(-\frac{i}{2} U \right) \delta_{b_1 - b_2} \left(-\frac{i}{2} U \right) \delta_{b_1 - b_2} \int \frac{d\omega_1}{2\pi} \int \frac{d^3 k_1}{(2\pi)^3} \times$$

$$\times (-1) i G_{b_1, b_2}^{c_1 0}(k_1, \omega_1) i G_{b_2, b_1}^{c_1 0}(k_1, \omega_1) N_0(\mu).$$

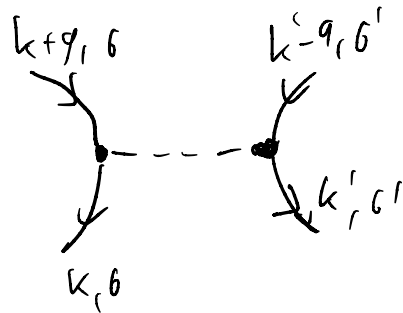
Analogous for polarization diagram.

2.) (i) 3D Jellium model in reciprocal space:

$$H = \sum_{k, \sigma} \epsilon_k c_{k, \sigma}^\dagger c_{k, \sigma} + \frac{1}{2N} \sum_{k, k', \sigma, \sigma'} \sum_{q \neq 0} V_q c_{k+q, \sigma}^\dagger c_{k'-q, \sigma'}^\dagger c_{k', \sigma'} c_{k, \sigma}$$

$\uparrow$
 $q=0$ cancelled
 by Jellium background

(spin conserved at every vertex)



$$\epsilon_k = \frac{k^2}{2m} - \epsilon_F, \quad \epsilon_F = \frac{k_F^2}{2m}, \quad V_q = \frac{1}{q^2} \quad (\text{F.T. of Coulomb vertex})$$

(ii) Dyson equation for polarization operator

$$\Pi = \Pi_0 + \Pi_0 \Pi$$

with $\Pi = i\Pi = \text{bubble diagram} + \mathcal{O}(u)$

$\uparrow$ $\uparrow$

$= i\Pi^{(0)}$

$$\sim \approx \approx \approx \approx \xrightarrow{\text{RPA}} \approx \text{---} + \text{---} \circ \text{---} + \text{---} \circ \text{---} \circ \text{---} + \dots$$

$$i\Pi^{(0)}(\underbrace{k, \omega}_P) = \text{Diagram: } \begin{array}{c} \text{P}_1 + P \\ \text{P} \quad \text{P}_1 \quad \text{P} \end{array} = \underset{\substack{\uparrow \\ \text{Spin} \\ \text{Summation}}}{2} \int \frac{d\omega_1}{2\pi} \int \frac{d^3k_1}{(2\pi)^3} \underset{\substack{\uparrow \\ \text{Fermi-loop}}}{(-1)} iG_c^0(k_1 + k, \omega + \omega_1) iG_c^0(k_1, \omega_1)$$

$$\left[\text{---} \text{---} \text{---} \text{---} \frac{1}{N} \sum_{k_1} \rightarrow \int \frac{d^3k_1}{(2\pi)^3} \text{---} \text{---} \text{---} \right]$$

$$\left[\begin{array}{l} \text{outside FS} \\ \downarrow \\ G_c^0(k, \omega) \underset{\substack{\uparrow \\ \text{Fermi-sphere} \\ (\text{FS})}}{=} \frac{\Theta(k - k_F)}{\omega - \epsilon_k + i0^+} + \frac{\Theta(k_F - k)}{\omega - \epsilon_k - i0^+} \end{array} \right]$$

inside FS ($\epsilon_k < 0$)

$$\leadsto G_c^0(k+k_1, \omega+\omega_1) G_c^0(k_1, \omega_1) = \frac{\theta(|k_1+k|-k_F) \theta(|k_1|-k_F)}{(\omega+\omega_1 - \epsilon_{k+k_1} + i0^+) (\omega_1 - \epsilon_{k_1} + i0^+)} \quad \leftarrow \text{both outside FS} \quad (I)$$

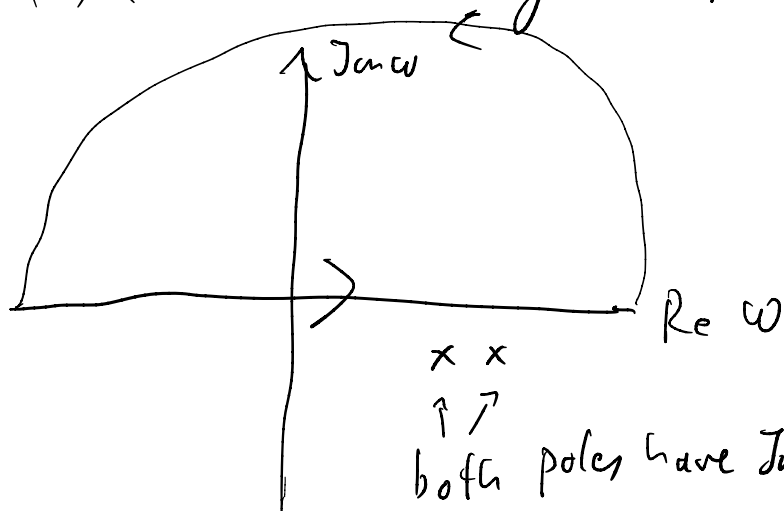
$$+ \frac{\theta(|k_1+k|-k_F) \theta(k_F - |k_1|)}{(\dots + i0^+) (\dots - i0^+)} \quad \leftarrow \text{one in one out} \quad (II)$$

$$+ \frac{\theta(k_F - |k_1+k|) \theta(|k_1| - k_F)}{(\dots - i0^+) (\dots + i0^+)} \quad \leftarrow \text{one in one out} \quad (III)$$

$$+ \frac{\theta(k_F - |k_1+k|) \theta(k_F - |k_1|)}{(\dots - i0^+) (\dots - i0^+)} \quad \leftarrow \text{both inside FS}$$

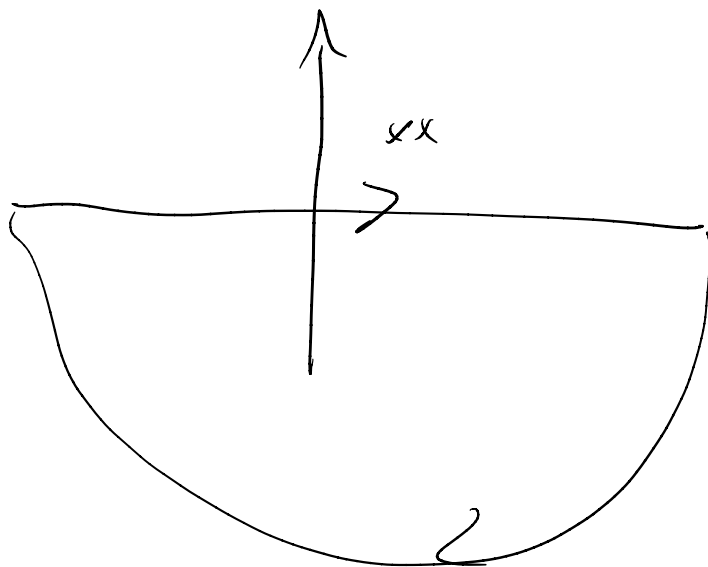
$\int \frac{dw}{2\pi} (I)-(IV)$ solved by complex integration using residue calculus

(I)



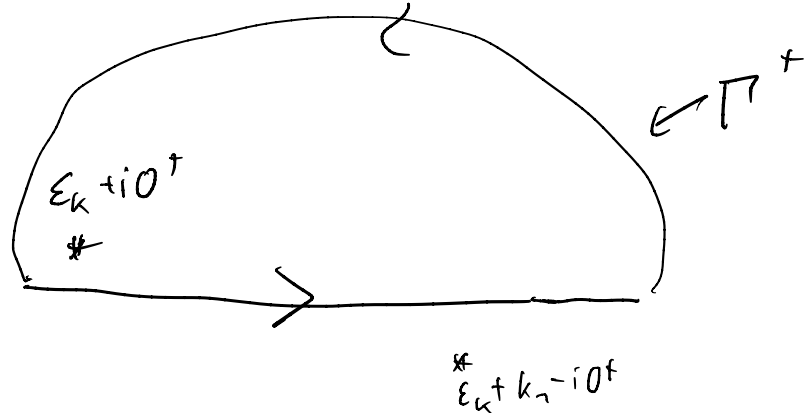
$$\leadsto \int dw (I) = 0$$

(IV)



$$\leadsto \int dw (IV) = 0$$

(II)



$$\oint_{\Gamma^+} d\omega_1 \Pi = 2\pi i \text{Res}(\Pi, \omega_1 = \epsilon_{k_1} + i0^+)$$

$$= 2\pi i \lim_{\omega_1 \rightarrow \epsilon_k + i0^+} (\omega_1 - \epsilon_{k_1} - i0^+) \cdot (\Pi)$$

$$= 2\pi i \frac{\theta(|k_1 + k| - k_F) \theta(k_F - k_1)}{\omega + \epsilon_{k_1} - \epsilon_{k_1 + k} + i0^+}$$

Analogously for (III)

For $\int (\text{III}) d^3 k_1$, use $k_1 \rightarrow -k_1$, and then $k_1 \rightarrow k_1 + k$

$$\leadsto i\pi^{(0)}(k, \omega) = 2i \int \frac{d^3 k_1}{(2\pi)^3} \theta(|k_1 + k| - k_F) \theta(k_F - |k_1|) \left[\frac{1}{\omega + \epsilon_{k_1} - \epsilon_{k_1 + k} + i0^+} - \frac{1}{\omega + \epsilon_{k_1 + k} - \epsilon_{k_1} - i0^+} \right]$$

$$\left[\log \left| 1 + \frac{k}{k_F - \frac{k}{2}} \right| \right] = \frac{k}{k_F} + \frac{1}{12} \left(\frac{k}{k_F} \right)^3 + \mathcal{O}(k^5)$$

$\leadsto$ Lindhard :

$$\Pi^{(0)}(k, 0) = - \frac{2m k_F}{(2\pi)^2} \left[1 + \frac{k_F^2 - \left(\frac{k}{2}\right)^2}{k_F k} \log \left| \frac{k_F + \frac{k}{2}}{k_F - \frac{k}{2}} \right| \right]$$

$$= - \frac{2m k_F}{(2\pi)^2} \left[1 + \left(1 - \frac{k^2}{4k_F^2} + \dots \right) \left(1 + \frac{1}{2} \left(\frac{k}{k_F} \right)^2 + \dots \right) \right] =$$

$$= - \frac{4m k_F}{(2\pi)^2} + \mathcal{O}(k^2)$$

$$V_{eff}(k, \omega=0) = V(k) \sum_{l=0}^{\infty} \left(\Pi^{(0)}(k, 0) V(k) \right)^l = \frac{V(k)}{1 - \Pi^{(0)}(k, 0) V(k)} =$$

$$= \frac{1}{k^2 + \Pi^{(0)}(k, 0)} = \frac{1}{k^2 + q_{TF}^2} \quad \text{with } q_{TF} = \frac{4mk_F}{2\pi}$$

FT to real space:

$$\begin{aligned}
 V_{\text{eff}}(r) &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{1}{k^2 + q_{\text{TF}}^2} \\
 &\stackrel{\substack{\text{azimuthal} \\ \text{trivial}}}{=} \frac{1}{(2\pi)^2} \int_{-1}^{+1} du \int_0^\infty dk e^{ikru} \frac{k^2}{k^2 + q_{\text{TF}}^2} \stackrel{u = \cos\theta}{=} \\
 &\stackrel{\substack{\text{perform} \\ u\text{-integral}}}{=} \frac{1}{(2\pi)^2} \frac{2}{r} \int_0^\infty dk \frac{k \sin(kr)}{k^2 + q_{\text{TF}}^2} = \frac{1}{(2\pi)^2} \frac{1}{r} \int_{-\infty}^{+\infty} dk \frac{k \sin(kr)}{k^2 + q_{\text{TF}}^2} \stackrel{\substack{\text{integrand} \\ \text{even in } k}}{=} \\
 &= \frac{1}{(2\pi)^2} \frac{1}{2ri} \left[\underbrace{\int_{-\infty}^{+\infty} dk \frac{k e^{ikr}}{k^2 + q_{\text{TF}}^2}}_{(1)} - \underbrace{\int_{-\infty}^{+\infty} dk \frac{k e^{-ikr}}{k^2 + q_{\text{TF}}^2}}_{(2)} \right]
 \end{aligned}$$

Use pole structure of $k^2 + q_{\text{TF}}^2 = (k + iq_{\text{TF}})(k - iq_{\text{TF}})$ to obtain

$$(1) = 2\pi i \operatorname{Res}\left(\frac{k e^{ikr}}{k^2 + q_{\text{TF}}^2}, iq_{\text{TF}}\right) = i\pi e^{-q_{\text{TF}} r}$$

$$(2) = 2\pi i \operatorname{Res}\left(\frac{k e^{-ikr}}{k^2 + q_{\text{TF}}^2}, -iq_{\text{TF}}\right) = -i\pi e^{-q_{\text{TF}} r}$$

$$\leadsto \frac{1}{(2\pi)^2} \frac{1}{2ri} [\textcircled{1} - \textcircled{2}] = V_{\text{eff}}(r) = \frac{1}{4\pi r} e^{-q_{\text{TF}} r}$$