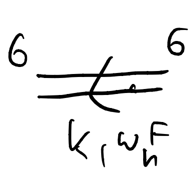
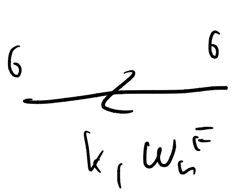
ATQ 2022/23 — Sheet 3

(a) • Recap of relevant diagrammatic decoding in Matsubara formalism:

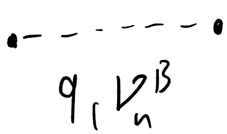
 $\simeq -\Pi^M$ (polarization operator)



$\simeq -G^M_{\sigma,\sigma'}(k, \omega_n^F)$ (full propagator)



$\simeq -G^{M,0}_{\sigma,\sigma'}(k, \omega_n^F) = \dfrac{\delta_{\sigma\sigma'}}{i\omega_n - \varepsilon_k}$ (free propagator)

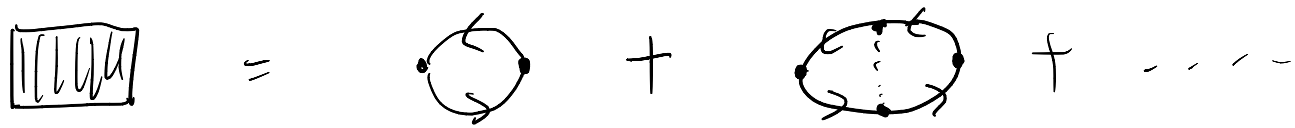


$\simeq -U(p, \nu_n^B)$ (Interaction, even Matsubara frequencies to facilitate conservation at every vertex).

$\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3} \cdots$ at every internal vertex.

(-1) for every Fermi loop with more than one vertex.

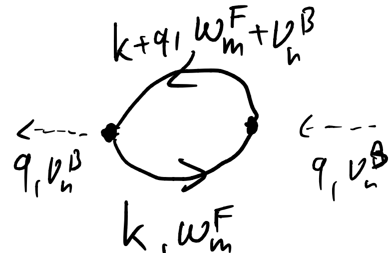
- Diagrammatic expansion of polarization



- Resummation for effective interaction

- Lowest order approximation to Matsubara polarization operator

$$-\Pi_0^M(q, \nu_n^B) =$$



independent dummy variables

$$\simeq (-1)\, 2\, \frac{1}{\beta} \sum_m \int \frac{d^3k}{(2\pi)^3}\, G^{M,0}(k, \omega_m^F) \times$$

spin

Fermi loop

$$\times\ G^{M,0}(k+q, \omega_m^F + \nu_n^B) =$$

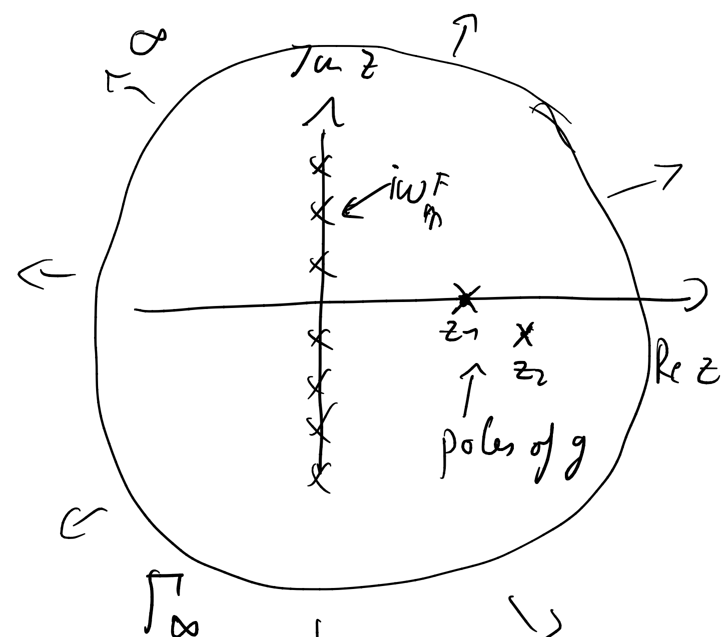
$$= \frac{-2}{\beta} \sum_m \int \frac{d^3k}{(2\pi)^3} \underbrace{\frac{1}{i\omega_m^F - \varepsilon_k} \; \frac{1}{i(\omega_m^F + \nu_n^B) - \varepsilon_{k+q}}}$$

$$=: g(\underbrace{i\omega_m^F}_{=:\, z}) \quad \left(\text{decays as } \frac{1}{|z|^2} \text{ for large } |z| \right)$$

- $g(z)$ has poles at z_ℓ, $\ell = 1,2$ with $z_1 = \varepsilon_k$, $z_2 = \varepsilon_{k+q} - i\nu_n^B$;

$\tanh\left(\frac{\beta z}{2}\right)$ has poles with residue $\frac{2}{\beta}$ at $z_m = i\omega_m^F$ (see Lecture 9)

$$0 = \oint_{\Gamma_\infty} \tanh\left(\frac{\beta z}{2}\right) g(z) \overset{\text{residue theorem}}{=}$$



$$= 2\pi i \, \frac{2}{\beta} \sum_m g(i\omega_m^F) + \sum_{\ell=1}^{2} 2\pi i \operatorname{Res}(g, z_\ell) \tanh\left(\frac{\beta z_\ell}{2}\right)$$

$$\rightsquigarrow \quad -\frac{2}{\beta} \sum_m g(i\omega_m^F) = \sum_{\ell=1}^{2} \tanh\left(\frac{\beta z_\ell}{2}\right) \operatorname{Res}(g, z_\ell)$$

With $\mathrm{Res}(g, z_1) = \dfrac{1}{i\nu_n^B + \varepsilon_k - \varepsilon_{k+q}}$, $\mathrm{Res}(g, z_2) = -\dfrac{1}{i\nu_n^B + \varepsilon_k - \varepsilon_{k+q}}$

and $\tanh\left(\beta \frac{w}{2}\right) = 1 - \dfrac{2}{e^{\beta w}+1} \rightsquigarrow \tanh\left(\frac{w + i\nu_n^B}{2}\beta\right) = 1 - \dfrac{2}{e^{w}\underbrace{e^{i\nu_n^B\beta}}_{e^{2\pi i n} = 1}+1} =$

$\left[n_F(\varepsilon_k) = \dfrac{1}{e^{\beta\varepsilon_k}+1} \right]$

$= \tanh\left(\frac{w}{2}\beta\right)$, we get

$$-\frac{2}{\beta}\sum_m g(i\omega_m^F) = \frac{\tanh\left(\frac{\varepsilon_k \beta}{2}\right) - \tanh\left(\varepsilon_{k+q}\frac{\beta}{2}\right)}{i\nu_n^B - (\varepsilon_{k+q} - \varepsilon_k)} = 2\,\frac{n_F(\varepsilon_{k+q}) - n_F(\varepsilon_k)}{i\nu_n^B - (\varepsilon_{k+q} - \varepsilon_k)}$$

(b) Compute $\Pi_0^M(q, \nu_n^B = 0)$ in the high-T/low-density limit

$$\Pi_0^M(q, 0) = 2\int \frac{d^3k}{(2\pi)^3}\, \frac{n_F(\varepsilon_k)\left(1 - n_F(\varepsilon_{k+q})\right) - n_F(\varepsilon_{k+q})\left(1 - n_F(\varepsilon_k)\right)}{\varepsilon_k - \varepsilon_{k+q}} =$$

[$k' = -k-q$ in second term]

[$\varepsilon_k = \varepsilon_{-k}$]

[insert $0 = n_F(\varepsilon_k) n_F(\varepsilon_{k+q}) - n_F(\varepsilon_k) n_F(\varepsilon_{k+q})$]

$$= 2\int \frac{d^3k}{(2\pi)^3} \frac{n_F(\varepsilon_k)(1-n_F(\varepsilon_{k+q}))}{\varepsilon_k - \varepsilon_{k+q}} - 2\int \frac{d^3k'}{(2\pi)^3} \frac{n_F(\varepsilon_{k'})(1-\varepsilon_{k'+q})}{\varepsilon_{k'+q} - \varepsilon_{k'}} =$$

rename $k' \to k$

$$= 2\int \frac{d^3k}{(2\pi)^3} n(\varepsilon_k)(1-n_F(\varepsilon_{k+q}))\left[\frac{1}{\varepsilon_k - \varepsilon_{k+q}} - \frac{1}{\varepsilon_{k+q} - \varepsilon_k}\right] =$$

$$4\int \frac{d^3k}{(2\pi)^3} n(\varepsilon_k)(1-n_F(\varepsilon_{k+q})) \frac{1}{-(\varepsilon_{k+q} - \varepsilon_k)} =$$

$$= 4\int \frac{d^3k}{(2\pi)^3} n_F(\varepsilon_k) \frac{1}{-(\varepsilon_{k+q} - \varepsilon_k)} - \cancelto{0}{\underbrace{4\int \frac{d^3k}{(2\pi)^3} \underbrace{n_F(\varepsilon_k)n_F(\varepsilon_{k+q})}_{\text{even in } k \leftrightarrow k+q} \underbrace{\frac{1}{\varepsilon_k - \varepsilon_{k+q}}}_{\text{odd in } k \leftrightarrow k+q}}_{\text{even in } k \leftrightarrow k+q \ \lightning}}$$

$$\overset{\boxed{n_F(\varepsilon_\alpha) = e^{-\beta\varepsilon_\alpha}}}{=} 4\int \frac{d^3k}{(2\pi)^3} e^{\beta\mu} e^{-\beta\frac{k^2}{2m}} \frac{1}{-\frac{q^2}{2m} - \frac{1}{m}|kq|\cos\Omega} =$$

$$\boxed{\varepsilon_{k+q} - \varepsilon_k = \frac{q^2}{2m} + \frac{1}{m}|kq|\cos\Omega}$$

$$\overset{\boxed{\text{azimuthal trivial}}}{=} -\frac{4}{(2\pi)^2} \int_0^\infty dk\, k^2 e^{\beta\mu} e^{-\beta\frac{k^2}{2m}} \int_{-1}^{+1} du \frac{1}{\frac{q^2}{2m} + \frac{kqu}{m}} =$$

$$= -\frac{8m}{(2\pi)^2} \frac{1}{q} e^{\beta\mu} \int_0^\infty dk\, k^2 e^{-\beta\frac{k^2}{2m}} \int_{-1}^{1} du \frac{1}{q+2ku} \overset{\boxed{x = 2ku}}{=}$$

$$= -\frac{4m}{(2\pi)^2} \frac{1}{q} e^{\beta\mu} \int_0^\infty dk\, k e^{-\beta\frac{k^2}{2m}} \int_{-2k}^{+2k} dx \frac{1}{q+x} =$$

$$= \frac{4m}{(2\pi)^2} \frac{1}{q} e^{-\beta\mu} \int_0^\infty dk\, k\, e^{-\beta\frac{k^2}{2m}} \log\left|\frac{q-2k}{q+2k}\right| =$$

$$\left[x = \lambda_{th} \cdot k \quad \text{with } \lambda_{th} = \sqrt{\frac{2\pi\beta}{m}} \right]$$

$$= -2\beta e^{\beta\mu} \lambda_{th}^{-3} f(\lambda_{th} q), \quad \text{where}$$

$$f(y) = \frac{1}{\pi y} \int_0^\infty dx\, x\, e^{-\frac{x^2}{4\pi}} \log\left|\frac{2x+y}{2x-y}\right| .$$

(c) Study $\lim_{q\to 0} \Pi_0^\mu(q,0)$

- $$\lim_{y\to 0} f(y) = \frac{1}{\pi} \int_0^\infty dx\, e^{-\frac{x^2}{4\pi}} = 1$$

$\log|1+w| = w + \ldots$

$$\left[\lim_{y\to 0} \frac{1}{y} \log\left|\frac{2x+y}{2x-y}\right| = \lim_{y\to 0} \frac{1}{y} \log\left|1 + \frac{2y}{2x-y}\right| = \frac{1}{x} \right]$$

- $\Pi_0^{\mu}(q \to 0, 0) = -2\beta\, e^{\beta\mu} \lambda_{th}^{-3} =: -q_{DH}^2$

$$q_{DH}^2 = 2\beta\, e^{\beta\mu} \left(\frac{m}{2\pi\beta}\right)^{\frac{3}{2}} = \beta n_0 \quad \text{with} \quad n_0 = 2\int \frac{d^3k}{(2\pi)^3} n_F(\varepsilon_k).$$

- Resummation to effective interaction:

$$V_{eff}(q,0) \simeq \frac{\frac{1}{q^2}}{1 + q_{DH}^2 \frac{1}{q^2}} = \frac{1}{q^2 + q_{DH}^2}.$$

FT to real space

[See solution Sheet 2]

$$\xrightarrow{\text{FT to real space}} \quad V_{eff}(x) = \int \frac{d^3q}{(2\pi)^3} e^{iqx} V(q,0) = \frac{1}{4\pi|x|} e^{-q_{DH}|x|}.$$