ATQ 2022/23 – Sheet 4

1. (a) $$I(V,t) \underset{\text{Def. current}}{=} -|e|\left\langle \frac{d}{dt} \underbrace{N_R}_{:=\sum_{q,\sigma} d^\dagger_{q,\sigma} d_{q,\sigma}}(t)\right\rangle \underset{\text{Heisenberg equation}}{=} -i|e|\langle [H, N_R](t)\rangle =$$

$$\underset{[H_L+H_R,\, N_R]=0}{=} -i|e|\langle [H_T, N_R](t)\rangle =$$

$$= -i|e| \sum_{\substack{q,k,\sigma \\ q',\sigma'}} \langle [T c^\dagger_{k,\sigma} d_{q,\sigma} + T^* d^\dagger_{q,\sigma} c_{k\sigma},\ d^\dagger_{q',\sigma'} d_{q',\sigma'}](t)\rangle =$$

$[c_{k\sigma},\ d^\dagger_{q',\sigma'} d_{q',\sigma'}] = 0$

$[d_{q,\sigma},\ d^\dagger_{q',\sigma'} d_{q',\sigma'}] = \delta_{q,q'}\delta_{\sigma,\sigma'} d_{q,\sigma}$

$$= -i|e| \sum_{k,q,\sigma} \left(T\langle c^\dagger_{k,\sigma}(t) d_{q,\sigma}(t)\rangle - T^*\langle d^\dagger_{q\sigma}(t) c_{k\sigma}(t)\rangle \right) =$$

$$= -i|e| \sum_{k,q,\sigma} \left(iT \left(F^{-+}_{k,q,\sigma}(0)\right)^* - T^*(-i) F^{-+}_{k,q,\sigma}(0) \right) =$$

$$\boxed{F^{-+}_{k,q,\sigma}(t) := i \langle d^+_{q,\sigma}(0)\, c_{k,\sigma}(t) \rangle}$$

$$= 2|e| \,\mathrm{Re}\left(T^* \sum_{k,q,\sigma} F^{-+}_{k,q,\sigma}(0) \right) \checkmark =$$

[perform trivial spin summation] $\longrightarrow$

$$= 4|e| \,\mathrm{Re}\left(T^* \sum_{k,q} F^{-+}_{k,q}(0) \right)$$

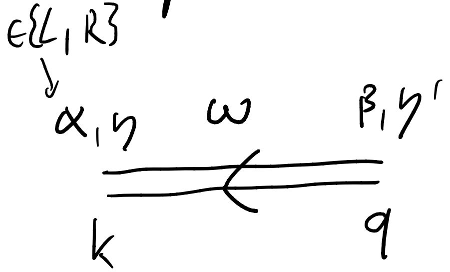
[Rest of calculation with spinless/spin-summed expressions]

(b) • Full structure of GF: $\hat{G} = \begin{pmatrix} \hat{G}_{LL} & \hat{G}_{LR} \\ \hat{G}_{RL} & \hat{G}_{RR} \end{pmatrix}$,

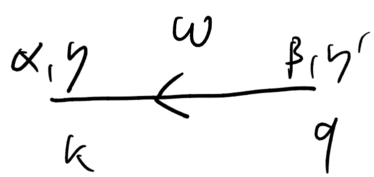
where $\hat{G}_{LR} = \hat{F} = \begin{pmatrix} F^{--} & F^{-+} \\ F^{+-} & F^{++} \end{pmatrix}$.

↳ H independent of time $\rightsquigarrow \hat{G}(t-t') \underset{\text{F.T.}}{\rightsquigarrow} \hat{G}(\omega)$ in frequency space.

- Diagrammatic translation table:

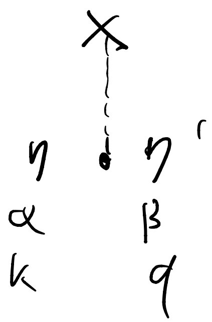


$$= i\, G^{\eta\eta'}_{\alpha\beta}(k, q, \omega)$$



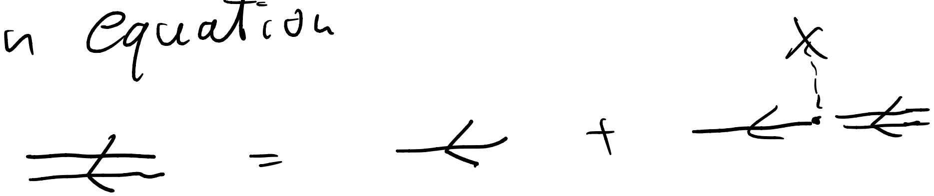
$$= i\, G^{\eta\eta'}_{0,\alpha\beta}(k, q, \omega) \sim \delta_{q,k}\, \delta_{\alpha,\beta}$$

[free dynamics conserves momentum and does not couple leads]



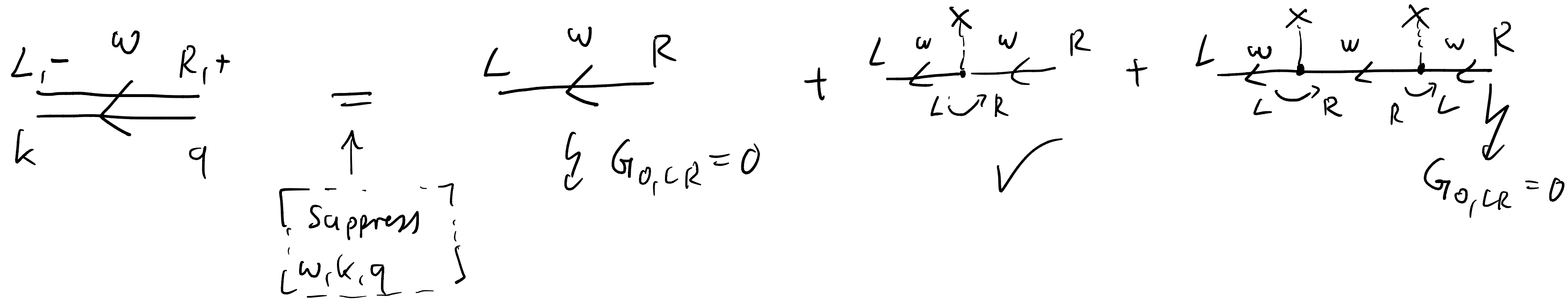
$$= -iV^{\eta\eta'}_{\alpha,\beta} = -i \underbrace{\eta\, \delta_{\eta,\eta'}}_{=\hat{\tau}_3} \left(T \delta_{\alpha L} \delta_{\beta R} + T^* \delta_{\alpha R} \delta_{\beta L} \right)$$

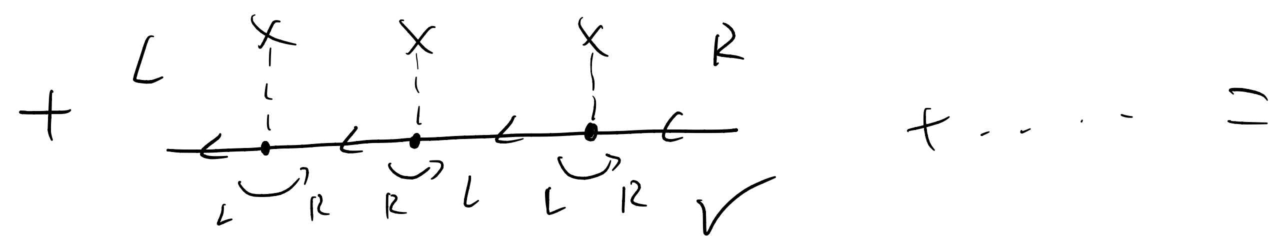
- Dyson equation



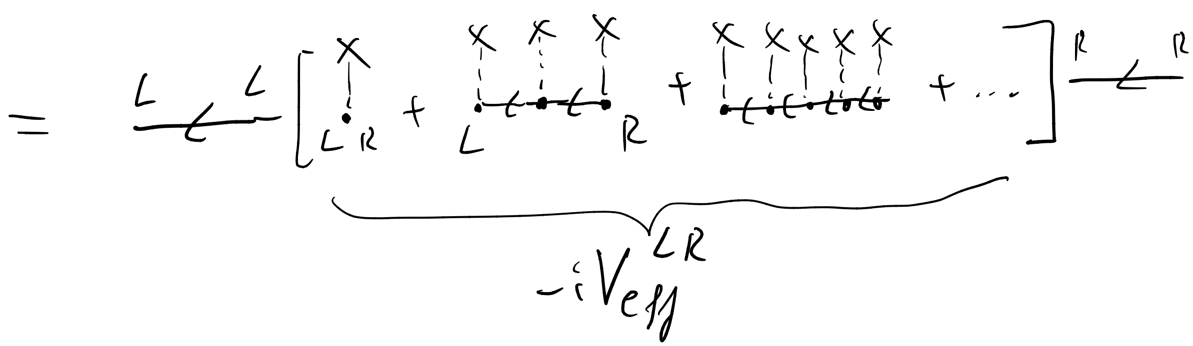
in reciprocal space.

- We want to solve for $G^{-+}_{LR}(k, q, \omega)$:





[only odd numbers of potential insertions can contribute due to off-diagonal structure of V and diagonal structure of G_0]



• Larkin-Ovchinnikov representation:

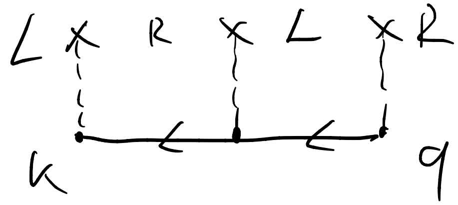
$$\bar G \overset{\text{Lec. 13}}{=} U \hat\tau_3 \hat G U^\dagger = \begin{pmatrix} G^R & G^K \\ & G^A \end{pmatrix}, \quad \text{where } U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}; \quad \hat\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$-i\bar V = -i U \tau_3 \hat V U^\dagger = U\Big(-i \underbrace{\hat\tau_3 \hat\tau_3}_{=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =: \hat\tau_0} \big(T\delta_{\alpha L}\delta_{\beta R} + T^*\delta_{\alpha R}\delta_{\beta L}\big)\Big)U^\dagger =$$

$$= -i\hat\tau_0 \big(T\delta_{\alpha L}\delta_{\beta R} + T^*\delta_{\alpha R}\delta_{\beta L}\big).$$

$\rightsquigarrow$ $\bar V$ is identity matrix in L.-O.-space and momentum-independent.

• From this structure, we obtain:



$$= \underbrace{-iT}_{-i\bar V_{LR}}\; i\underbrace{\sum_{k_1} \bar G_{0,RR}(k_1,\omega)}_{=: \bar g_R(\omega)}\; \underbrace{-iT^*}_{-i\bar V_{RL}}\; \bar g_L(\omega)\; \underbrace{-iT}_{-i\bar V_{LR}} = -iT|T|^2 \bar g_R \bar g_L,$$

and thus $-i\bar V^{LR}_{eff} = -iT\big[1 + |T|^2 \bar g_R \bar g_L + |T|^4 (\bar g_R \bar g_L)^2 + \dots\big] =$

$$= -iT \sum_{\ell=0}^{\infty} \left(|T|^2 \bar{g}_R \bar{g}_L \right)^\ell \underset{\uparrow}{=} -iT \left(1 - |T|^2 \bar{g}_R \bar{g}_L \right)^{-1}$$

[geometric series]

$$\bar{g}_R \bar{g}_L = \begin{pmatrix} g_R^R & g_R^K \\ 0 & g_R^A \end{pmatrix} \begin{pmatrix} g_L^R & g_L^K \\ 0 & g_L^A \end{pmatrix} = \begin{pmatrix} g_R^R g_L^R & g_R^R g_L^K + g_R^K g_L^A \\ 0 & g_R^A g_L^A \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

we get

$$-i\bar{V}_{eff}^{LR} = \frac{-iT}{\underbrace{\left(1-|T|^2 g_R^R g_L^R\right)\left(1-|T|^2 g_R^A g_L^A\right)}_{=: a\cdot d}} \begin{pmatrix} 1-|T|^2 g_R^A g_L^A & -|T|^2 \left(g_R^R g_L^K + g_R^K g_L^A\right) \\ 0 & 1-|T|^2 g_R^R g_L^R \end{pmatrix}.$$

$$i \sum_{k,q} \bar{G}_{LR}(k,q,\omega) = i \sum_k \bar{G}_{0,LL}(k,\omega) \left(-i\bar{V}_{eff}^{LR}(\omega)\right) i \sum_q \bar{G}_{0,RR}(q,\omega) = i\bar{g}_L (-i) \bar{V}_{eff}^{LR} \, i\bar{g}_R =$$

$$= \frac{iT}{a\cdot d} \begin{pmatrix} g_R^R & g_R^K \\ & g_R^A \end{pmatrix} \begin{pmatrix} \quad - \text{"} - \quad \end{pmatrix} \begin{pmatrix} g_L^R & g_L^K \\ 0 & g_L^A \end{pmatrix} =$$

[Langreth rules Lec. 14]

$$= \frac{iT}{a\cdot d} \begin{pmatrix} g_R^R g_L^R \left(1-|T|^2 g_R^A g_L^A\right) & \tilde{g}_{LR}^K \\ 0 & g_R^A g_L^A \left[1-|T|^2 g_R^R g_L^R\right] \end{pmatrix}$$

with $\tilde{g}^K_{CR} = g^R_R\left(1-|T|^2 g^A_R g^A_L\right) g^K_L + g^R_R\left(-|T|^2\left[g^R_R g^K_L + g^K_R g^A_L\right]\right) g^A_L + g^K_R\left(1-|T|^2 g^R_R g^R_L\right) g^A_C$

↳ Trasform back from $\bar{G}$ to $\hat{G}$, concretely $G^{-+} = \frac{1}{2}\left(G^A - G^R + G^K\right)$,

and thus $i\sum_{k,q} G^{-+}(k,q,\omega) \equiv i\sum_{k,q} F^{-+}_{k,q}(\omega) =$

[Time to plug in some "numbers"]

$g^{R/A}_{L/R}(\omega) \approx \mp i\pi \rho_{L/R}(\omega + \mu_{L/R})$

$g^K_{L/R}(\omega) \approx -2\pi i\left[1-2n_F(\omega+\mu_{L/R})\right]\rho_{L/R}(\omega+\mu_{L/R})$

[$n_F^{L/R} = n_F(\omega + \mu_{L/R})$]

straight-forward algebra

$$= \cdots = \frac{iT\, 2\pi^2 \rho_R \rho_L \left(n_F^L - n_F^R\right)}{\left(1 + \pi^2 |T|^2 \rho_R \rho_L\right)^2} .$$

$$\rightsquigarrow I(V,t) \underset{(a)}{=} 4|e|\int \frac{d\omega}{2\pi} \mathrm{Re}\left[T^* \sum_{k,q} F^{-+}_{k,q}(\omega)\right] = 4|e|\int \frac{d\omega}{2\pi} \frac{|T|^2 2\pi^2 \rho_L \rho_R}{\left[1+\pi^2|T|^2\rho_L\rho_R\right]^2}\left(n_F^L - n_F^R\right)$$

$$= 4|e| \int_{-\mu_R}^{-\mu_L} \frac{d\omega}{2\pi} \frac{|T|^2 2\pi^2 \rho_L \rho_R}{[1+\pi^2|T|^2 \rho_L \rho_R]^2} \overset{\omega \to \omega+\mu_R}{=} 4|e|\, |T|^2 2\pi^2 \int_0^{\mu_R-\mu_L=|e|V} \frac{d\omega}{2\pi} \frac{\rho_L(\omega-|e|V)\,\rho_R(\omega)}{[1+\pi^2|T|^2\rho_L(\omega-|e|V)\,\rho_R(\omega)]^2} \checkmark$$

$$\left[n_F^L - n_F^R = \big(1-\theta(\omega+\mu_L)\big) - \big(1-\theta(\omega+\mu_R)\big) \overset{\mu_R>\mu_L}{=} \begin{cases} 1 & -\mu_R < \omega < -\mu_L \\ 0 & \text{otherwise} \end{cases} \right]$$

(c) Limit of small V: $G = \dfrac{I}{V} \approx \dfrac{4|e|^2 |T|^2 2\pi^2 \frac{\cancel{V}}{2\pi} \rho_L(0)\rho_R(0)}{[1+\pi^2|T|^2\rho_L(0)\rho_R(0)]^2 \cancel{V}} =$

$$\left[G_0 = \frac{2e^2}{h} \overset{\hbar=1}{=} \frac{e^2}{\pi} \right]$$

$$= G_0 \cdot \frac{T_{eff}^2}{\left(1+\frac{T_{eff}^2}{4}\right)^2}$$

$$\left[T_{eff}^2 = \big(2\pi|T|\rho_L(0)\big)\big(2\pi|T|\rho_R(0)\big) \right]$$