ATQ 2022/23 – Sheet 5

1) (i) $$\rho_r(x) = c_r^\dagger(x)\, c_r(x) = \frac{1}{L}\sum_{p,q} e^{i(q-p)x}\, c_{p,r}^\dagger c_{q,r} \underset{k=q-p}{=} \sum_{k,q} e^{ikx} c_{q-k,r}^\dagger c_{q,r}$$

$$\underset{\boxed{\rho_{k,r} = \sum_q c_{q-k,r}^\dagger c_{q,r}}}{=} \frac{1}{L}\sum_k e^{ikx}\rho_{k,r}.$$ From $\rho_r(x) = \rho_r^\dagger(x)$, we get

$$\rho_{k,r}^\dagger = \rho_{-k,r} = \sum_q c_{q+k,r}^\dagger c_{q,r}$$

(ii) Compute commutator

obvious

drop fixed r for brevity

$$[\rho_{k,r}, \rho_{k',r'}^\dagger] = \delta_{r,r'}\sum_{q,q'}[c_{q-k,r}^\dagger c_{q,r}, c_{q'+k',r}^\dagger c_{q',r}] =$$

$$= \delta_{r,r'}\sum_{q,q'}\left(c_{q-k}^\dagger c_q\, c_{q'+k'}^\dagger c_{q'} - c_{q'+k'}^\dagger c_{q'}\, c_{k-q}^\dagger c_q\right) =$$

$$\delta_{r,r'} \sum_{q,q'} \left(\cancel{- c^\dagger_{q-k} c^\dagger_{q'+k'} c_q c_{q'}} + \delta_{q,q'+k'} c^\dagger_{q-k} c_{q'} + \cancel{c^\dagger_{q'+k'} c^\dagger_{k-q} c_{q'} c_q} - \delta_{q',k-q} c^\dagger_{q'+k'} c_q \right)$$

$$= \delta_{r,r'} \sum_q \left(c^\dagger_{q-k} c_{q-k'} - c^\dagger_{q+k'-k} c_q \right).$$

[$q+k' \to q$ in second term]

- Case on $k \neq k'$: $\sum_q \left(c^\dagger_{q-k} c_{q-k'} - c^\dagger_{q+k'-k} c_q \right) = \sum_q \left(c^\dagger_{q-k} c_{q-k'} - c^\dagger_{q-k} c_{q-k'} \right) = 0$

[Eligible since vacuum expectation is zero for $k \neq k'$]

$$\rightsquigarrow [\rho_{k,r}, \rho^\dagger_{k',r'}] = \delta_{r,r'} \delta_{k,k'} \sum_q c^\dagger_{q-k} c_{q-k} - c^\dagger_q c_q = \delta_{r,r'} \delta_{k,k'} \sum_q (n_{q-k,r} - n_{q,r})$$

↳ Problem: infinite vacuum contribution, i.e. shift $q+k \to q$ not allowed!

↳ Solution: Consider normal ordered expressions such as

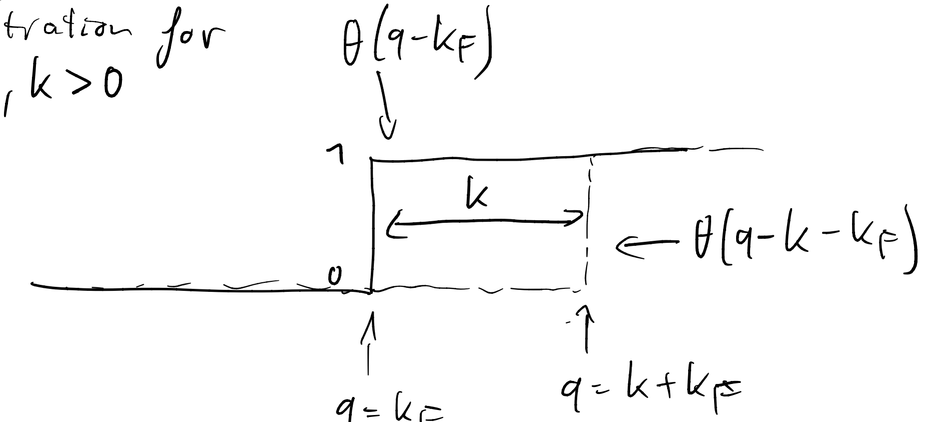
$$:n_{q,r}: = n_{q,r} - \langle n_{q,r} \rangle_0 = n_{q,r} - (1 - \theta(rq - k_F)).$$

$$\rightsquigarrow [\rho_{k,r}, \rho^\dagger_{k',r'}] = \delta_{r,r'}\delta_{k,k'} \sum_q \left[:n_{q-k,r}: - :n_{q,r}: + \theta(rq-k_F) - \theta(r(q-k)-k_F) \right] =$$

$$= \delta_{r,r'}\delta_{k,k'} \underbrace{\sum_q \theta(rq-k_F) - \theta(r(q-k)-k_F)}_{\pm \#\text{momenta } \frac{k}{2\pi/L} \text{ between } q \text{ and } q-k} = \delta_{r,r'}\delta_{k,k'}\, r \frac{Lk}{2\pi} \quad ✓$$

shift $q+k \to q$ eligible in normal ordered part

Illustration for $r=+$, $k>0$



Similar for other cases $r = \pm$, $k \gtrless 0$

2.) Consider bilinear Hamiltonian $H = \sum_j \varepsilon_j (b_j^\dagger b_j + \frac{1}{2}) \equiv \sum_j H_j$ and a linear operator $B = \sum_j (\alpha_j b_j^\dagger + \gamma_j b_j) \equiv \sum_j B_j$ expanded in the eigenbasis of H.

• We can use $\langle e^{B} \rangle \overset{[B_j, B_{j'}] = 0,\ j \neq j'}{=} \langle \prod_j e^{B_j} \rangle = \prod_j \langle e^{B_j} \rangle$

$[H_j, H_{j'}] \underset{j \neq j'}{=} 0$, $Z = \prod_j Z_j$ with

$$Z_j = \sum_{n_j=0}^{\infty} e^{-\beta \varepsilon_j (n_j + \frac{1}{2})} = \frac{e^{-\beta \frac{\varepsilon_j}{2}}}{1 - e^{-\beta \varepsilon_j}}$$

• Furthermore, $\langle B^2 \rangle \overset{\text{def. } B}{=} \sum_{j,j'} \left(\alpha_j \gamma_{j'} \langle b_j^{+} b_{j'} \rangle + \alpha_{j'} \gamma_j \underbrace{\langle b_j b_{j'}^{+} \rangle}_{= \langle b_{j'}^{+} b_j \rangle + \delta_{j,j'}} \right) =$

$\langle b_j b_{j'} \rangle = \langle b_j^{+} b_{j'}^{+} \rangle = 0$

$\langle b_j^{+} b_{j'} \rangle = \delta_{j,j'} \langle b_j^{+} b_j \rangle$

$$= 2 \sum_j \alpha_j \gamma_j \left(\langle b_j^{+} b_j \rangle + \frac{1}{2} \right), \text{ i.e.}$$

$$e^{\frac{\langle B^2 \rangle}{2}} = \prod_j e^{\alpha_j \gamma_j (\langle b_j^{+} b_j \rangle + \frac{1}{2})}$$

, thus to prove $\langle e^{B} \rangle = e^{\frac{\langle B^2 \rangle}{2}}$, if is sufficient to prove $\langle e^{B_j} \rangle = e^{\alpha_j \gamma_j \left(\langle b_j^\dagger b_j \rangle + \frac{1}{2} \right)}$

- $$\langle e^{B_j} \rangle = \frac{e^{\frac{\alpha\gamma}{2}}}{Z_j} \sum_{n=0}^{\infty} e^{-\beta \varepsilon (n+\frac{1}{2})} \underbrace{\langle n | e^{\alpha b^\dagger} e^{\gamma b} | n \rangle}_{} =$$

$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$

Drop j where unambiguous

$$\sum_{l,m=0}^{\infty} \frac{\alpha^l}{l!} \frac{\gamma^m}{m!} \langle n | (b^\dagger)^l (b)^m | n \rangle = \sum_{l=0}^{\infty} \frac{(\alpha\gamma)^l}{(l!)^2} \langle n | (b^\dagger)^l (b)^l | n \rangle =$$

$\sim \delta_{l,m}$

$$b^l |n\rangle = \begin{cases} \sqrt{\frac{n!}{(n-l)!}} \, |n-l\rangle & l \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{l=0}^{n} \frac{(\alpha\gamma)^l}{l!} \frac{n!}{(n-l)!\,l!} = \sum_{l=0}^{n} \frac{(\alpha\gamma)^l}{l!} \binom{n}{l}$$

$$= \frac{e^{\frac{\alpha\gamma}{2}}}{Z_j} \sum_{n=0}^{\infty} e^{-\beta\varepsilon(n+\frac{1}{2})} \sum_{l=0}^{n} \frac{(\alpha\gamma)^l}{l!} \binom{n}{l} = \frac{e^{\frac{\alpha\gamma}{2}}}{Z_j} \sum_{l=0}^{\infty} \frac{(\alpha\gamma)^l}{l!} \sum_{n=l}^{\infty} e^{-\beta\varepsilon(n+\frac{1}{2})} \binom{n}{l} =$$

$$\sum_{n=0}^{\infty} \sum_{l=0}^{n} \ldots = \sum_{0 \leq l \leq n < \infty} \ldots = \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} \ldots$$

$$\sum_{n=\ell}^{\infty} e^{-\beta\varepsilon(n+\frac{1}{2})} \binom{n}{\ell} \overset{m=n-\ell}{=} e^{-\beta\varepsilon(\ell+\frac{1}{2})} \sum_{m=0}^{\infty} e^{-\beta\varepsilon m} \binom{m+\ell}{\ell} = \frac{e^{-\beta\varepsilon(\ell+\frac{1}{2})}}{(1-e^{-\beta\varepsilon})^{\ell+1}} = \frac{e^{-\beta\varepsilon/2}}{1-e^{-\beta\varepsilon}} \left(\frac{1}{e^{\beta\varepsilon}-1}\right)^{\ell} =$$

[Def. Bose-Einstein distr. and def. Z_j]

$$= Z_j \langle b_j^+ b_j \rangle^{\ell}$$

[negative Binomial series

$$(x+a)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k a^{-n-k} \quad \text{for } |x|<a$$

i.e. $$(1-x)^{-(n+1)} = \sum_{k=0}^{\infty} \binom{n+k}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k$$]

i.e. $$\langle e^{B_j} \rangle = e^{\frac{\alpha_j \gamma_j}{2}} \sum_{\ell=0}^{\infty} \frac{(\alpha_j \gamma_j)^{\ell}}{\ell!} \langle b_j^+ b_j \rangle^{\ell} = e^{\alpha_j \gamma_j (\langle b_j^+ b_j \rangle + \frac{1}{2})} . \square$$