ATQ 2022/23 – Sheet 6

(a) 1D continuity equation $\quad e\,\dfrac{\partial \rho(x,t)}{\partial t} = -\partial_x J(x,t)$

With $\rho(x,t) \approx \rho_R(x,t) + \rho_L(x,t) = \dfrac{1}{\sqrt{\pi}}\partial_x \phi(x,t)$, we get:

$$-\partial_x\left(-\frac{e}{\sqrt{\pi}}\partial_t \phi(x,t)\right) = -\partial_x J(x,t)$$

$$\rightsquigarrow \quad J(x,t) = -\frac{e}{\sqrt{\pi}}\partial_t \phi(x,t) \qquad \text{(up to integration constant)}$$

✓

[no zero-order equilibrium current]

(b) Fourier transform $\langle J(x)\rangle(t) = \cancel{\langle J(x)\rangle_0(t)} + \int dx' \int dt'\, G^R_{JJ}(x-x',t-t')\,A(x,t)$

with $G^R_{JJ}(x-x',t-t') = -i\,\theta(t-t')\langle[J(x,t),J(x',t')]\rangle_0 \underset{(a)}{=} -\dfrac{ie^2}{\pi}\theta(t-t')\langle[\partial_t\phi(x,t),\partial_{t'}\phi(x',t')]\rangle_0$

$$\langle J(x)\rangle(\omega) = \int dt\, e^{i\omega t}\langle J(x)\rangle(t) = \int dt\, e^{i\omega t}\int dx' \int dt'\, G^R_{JJ}(x-x',t-t')\,A(x',t') =$$

$$= \int dx' \, G^R_{jj}(x-x', \omega) A(\omega) = \int dx' \, G^R_{jj}(x-x', \omega) \frac{-i E(\omega)}{\omega} \quad \checkmark$$

$(G * A)(t)$ (time-convolution)

$\rightsquigarrow (G * A)(\omega) = G(\omega) A(\omega)$

convolution theorem

$E(t) = -\partial_t A(t)$

$$\rightsquigarrow \frac{1}{2\pi}\int e^{-i\omega t} E(\omega)\, d\omega = -\partial_t \frac{1}{2\pi}\int e^{-i\omega t} A(\omega)\, d\omega$$

$$\rightsquigarrow \frac{1}{2\pi}\int e^{-i\omega t}\left[E(\omega)\right] d\omega = \frac{1}{2\pi}\int e^{-i\omega t}\left[i\omega A(\omega)\right] d\omega$$

$$\rightsquigarrow A(\omega) = \frac{-i}{\omega} E(\omega)$$

(c)

$$Z = \mathrm{tr}\left(e^{-\beta H}\right) = \int_{(\phi,\theta)(\tau=0) = (\phi,\theta)(\tau=\beta)} D\phi\, D\theta\, e^{-\int_0^\beta d\tau \int dx \left[-i\partial_x\theta\, \partial_\tau\phi + H_0(\phi,\theta)\right]},$$

$$H_0(\phi,\theta) = \frac{u}{2}\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right]$$

Think of $\int D\phi$ as $\lim_{N_1, N_2 \to \infty} \int \left(\prod_{j_1=0}^{N_1} \prod_{j_2=0}^{N_2} d\phi(x_{j_1}, \tau_{j_2})\right)$, similar for $D\theta$

(See Altland and Simons Chapter 3 + 4 for a pedagogical introduction)

[In reciprocal space $\int D\phi \to \int \prod_{q,n} d\phi_{q,i\omega_n}$]

q,n discrete for finite L

- Eliminate θ by Gaussian integration or using the Heisenberg equations of motion $\partial_\tau \phi(x,\tau) = [H,\phi] = \int dx' \frac{\partial H}{\partial(\partial_{x'}\theta)} [\partial_{x'}\theta(x',\tau), \phi(x,\tau)]$

$= -iuK\,\partial_x \theta(x,\tau)$, i.e. $-i\partial_x\theta = \frac{1}{uK}\partial_\tau \phi(x,\tau)$, and thus

[$[\phi(x), \partial_{x'}\theta(x')] = i\delta(x-x')$]

$$-i\partial_x\theta\,\partial_\tau\phi - H_0(\phi,\theta) = \frac{u}{2K}\left(\frac{1}{u^2}(\partial_\tau\phi)^2 + (\partial_x\phi)^2\right) \quad (11)$$

✓

- To reciprocal space: $\phi(x,\tau) = \sum_{q,i\omega_n} e^{i(qx-\omega_n\tau)}\phi_{q,i\omega_n}$

$$\rightsquigarrow \int d\tau \int dx\, \frac{u}{2K}\left(\frac{1}{u^2}(\partial_\tau\phi)^2 + (\partial_x\phi)^2\right) = \beta L \frac{u}{2K}\sum_{q,i\omega_n}\left(q^2 + \frac{\omega_n^2}{u^2}\right)\phi_{-q,-i\omega_n}\phi_{q,i\omega_n}.$$

[$\int_0^\beta d\tau\, e^{i(\omega_n+\omega_m)\tau} = \beta\delta_{\omega_n,-\omega_m}$; $\int_0^L dx\, e^{i(q+q')x} = L\delta_{q,-q'}$]

- Compute $\langle \phi_{-q,-i\omega_n} \phi_{q',i\omega_{n'}} \rangle_0 = \frac{1}{Z} \int \left(\prod_{\tilde{q},i\omega_m} d\phi_{\tilde{q},i\omega_m} \right) e^{-\frac{\beta L u}{2K} \sum_{\tilde{q},i\omega_m} \left(\tilde{q} + \frac{\omega_m^2}{u^2} \right) \phi_{-\tilde{q},-i\omega_m} \phi_{\tilde{q},i\omega_m}} \cdot \phi_{-q,-i\omega_n} \phi_{q',i\omega_{n'}} =$

Gaussian integration rule:

$$\frac{\int \left(\prod_j du_j^* du_j \right) e^{-\frac{1}{2} \vec{u}^\dagger A \vec{u}} u_x^* u_y}{\int \left(\prod_j du_j^* du_j \right) e^{-\frac{1}{2} \vec{u}^\dagger A \vec{u}}} = \left(A^{-1} \right)_{x,y}$$

only $\phi_{q>0,\omega_n>0}$ independent complex variables, since $\phi(x,\tau) \in \mathbb{R}$.
Here, $u_x^* \to \phi_{-q,-i\omega_n}$, $u_y \to \phi_{q,i\omega_n}$

$A_{xy} \to \delta_{q,q'} \delta_{n,n'} \frac{L\beta u}{K} \left(q^2 + \frac{\omega_n^2}{u^2} \right)$

$$= \delta_{q,q'} \delta_{n,n'} \frac{K}{L\beta u} \frac{1}{q^2 + \frac{\omega_n^2}{u^2}} \quad \checkmark \quad (*)$$

- With that compute current-current correlator:

$$\langle \tau_M \left(\partial_\tau \phi(x,\tau) \partial_{\tau'}(x',\tau') \right) \Big|_{\tau'=0} \rangle = \langle \tau_M \left(\sum_{q,i\omega_n} (-i\omega_n) \phi_{q,i\omega_n} e^{i(qx - \omega_n \tau)} \sum_{q',i\omega_{n'}} (-i\omega_{n'}) e^{iq'x'} \phi_{q',i\omega_{n'}} \right) \rangle_0$$

$$= \sum_{q, i\omega_n} (\omega_n^2)\, e^{i[q(x-x') - \omega_n \tau]} \frac{K}{\beta L u} \frac{1}{q^2 + \frac{\omega_n^2}{u^2}}$$

(*)

$$\begin{bmatrix} n = -n' \\ q' = -q \end{bmatrix}$$

$$\int d\tau\, e^{i(\omega_n - \omega_{n'})\tau} = \beta \delta_{n,n'}$$

$$\rightsquigarrow G^M_{jj}(x-x', i\omega_n) = -\frac{e^2}{\pi} \int d\tau\, e^{i\omega_n \tau} \left\langle T_M \left(\partial_\tau \phi(x,\tau)\, \partial_{\tau'} (x', \tau') \big|_{\tau'=0} \right) \right\rangle =$$

$$= -\frac{e^2}{\pi} \sum_q (\omega_n)^2 e^{iq(x-x')} \frac{K}{Lu} \frac{1}{q^2 + \frac{\omega_n^2}{u^2}} = -\frac{e^2}{\pi} \frac{K\omega_n^2}{u} \int \frac{dq}{2\pi} e^{iq(x-x')} \frac{1}{q^2 + \frac{\omega_n^2}{u^2}}$$

$$\left[\sum_q \to \frac{L}{2\pi} \int dq \right]$$

$$= -\frac{e^2 K \omega_n}{2\pi} e^{-|x-x'|\frac{\omega_n}{u}} \quad \checkmark$$

$$\left[\int \frac{dq}{2\pi} e^{iq(x-x')} \frac{1}{q^2 + a^2} = \frac{1}{2a} e^{-a|x-x'|} \right]$$

(d) Analytic continuation:

$$G^R_{JJ}(x-x',\omega) = G^M_{JJ}(x-x', i\omega_n \to \omega + i0^+) \overset{(c)}{=} -\frac{e^2 K(\omega+i0^+)}{i\, 2\pi}\, e^{i\frac{\omega+i0^+}{u}|x-x'|}.$$

Plug back into Kubo formula:

$\boxed{\text{DC-limit } \omega\to 0 \text{, uniform } E}$

$$\langle J(x)\rangle(\omega) = \int dx'\, E(x',\omega)\, \frac{\cancel{i}}{\cancel{\omega}}\, \frac{e^2 K \cancel{\omega}}{2\pi \cancel{i}}\, e^{i\frac{\omega}{u}|x-x'|} =$$

$$= \underbrace{E\, \frac{e^2 K}{2\pi} L}_{= E\sigma} \quad ; \quad G = \frac{\sigma}{L} = \frac{e^2}{2\pi} K \quad \checkmark$$