Open quantum systems – lecture 7

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Chapter 2

Stationary Quantum transport

The most obvious way to achieve non-equilibrium quantum dynamics even at steady state is to use a reservoir that is itself in a non-equilibrium state, i.e., a state that cannot simply be characterized by just one temperature and one chemical potential. Since the derivation of the master equation only requires $[\bar{\rho}_{\mathrm{B}}, H_B] = 0$, this would still allow for many nontrivial models, $\langle n| \bar{\rho}_{\mathrm{B}} |n\rangle$ could e.g. follow multi-modal distributions. Such a non-equilibrium situation may be established when a system is coupled to many different baths ν

$$H_B = \sum_{\nu=1}^{K} H_B^\nu \tag{2.1}$$

with commuting individual parts $\left[H_B^\ell, H_B^k\right] = 0$. The reservoirs are kept in local equilibrium states

$$\bar{\rho}_{\mathrm{B}} = \bigotimes_{\nu=1}^{K} \bar{\rho}_{\mathrm{B}}^\nu \qquad : \qquad \bar{\rho}_{\mathrm{B}}^\nu = \frac{e^{-\beta_\nu (H_B^\nu - \mu_\nu N_B^\nu)}}{Z_B^\nu}\,, \tag{2.2}$$

characterized by inverse temperature β_ν and chemical potential μ_ν and with N_B^ν denoting the particle number operator of reservoir ν, respectively. The **partition function** $Z_B^\nu = \mathrm{Tr}\left\{e^{-\beta_\nu (H_B^\nu - \mu_\nu N_B^\nu)}\right\}$ normalizes each reservoir density matrix.

Here, we will consider the case of such multiple reservoirs at different thermal equilibria that are only indirectly coupled via the system: Without the system, they would be completely independent and thus remain at their local equilibrium state. When coupled via the system, the reservoirs will exchange energy and thereby also their state will change in reality, as a battery is discharged in time. Within a master equation treatment, the change of the reservoirs however is not captured, which implies that there is already some assumption of the size difference between the system and the reservoirs. On the other hand, stationary heat and matter currents can be experimentally maintained over very large time scales, such that the assumption of a stationary reservoir can be very well justified in many setups.

Since these are chosen at different equilibria, they drag the system towards different thermal states, and the resulting stationary state is in general a non-thermal one. Since the different compartments interact only indirectly via the system, we have the case of a multi-terminal system, where one can most easily derive the corresponding master equation, since for weak couplings, each contact may be treated separately, as we shall see later.

To each of the reservoirs, the system is coupled via different coupling operators

$$H_I = \sum_\alpha S_\alpha \otimes B_\alpha = \sum_\alpha S_\alpha \otimes \sum_{\nu=1}^{K} B_\alpha^\nu \,. \tag{2.3}$$

Here, the index ν labels the sub-reservoir ν and B_α^ν acts non-trivally only on the Hilbert space of this sub-reservoir. Different coupling strengths are encoded in the B_α^ν: If for example one reservoir does not couple to the system operator A_α, we simply have $B_\alpha^\nu = 0$. Since we assume the setting that the first order bath correlation functions vanish $\langle B_\alpha^\nu \bar{\rho}_\mathrm{B} \rangle = 0$, the second-order bath correlation functions may be computed additively. In particular, we get from the tensor structure of the reservoir density matrix

$$\mathrm{Tr}\left\{ \boldsymbol{B_\alpha^\nu}(\tau) B_\beta^\mu \bar{\rho}_\mathrm{B} \right\} = \delta_{\mu\nu} \mathrm{Tr}\left\{ \boldsymbol{B_\alpha^\nu}(\tau) B_\beta^\nu \bar{\rho}_\mathrm{B}^\nu \right\} \,. \tag{2.4}$$

In other words, there is no correlation function between different reservoirs, and all relevant correlation functions can be computed by just considering the reservoirs separately. One should note that this is a weak-coupling statement: Going to higher than second order in the system-reservoir interaction strength will surely induce correlations between the reservoirs. We can thus write

$$C_{\alpha\beta}(\tau) = \sum_{\nu=1}^{K} \mathrm{Tr}\left\{ \boldsymbol{B_\alpha^\nu}(\tau) B_\beta^\nu \bar{\rho}_\mathrm{B}^\nu \right\} \equiv \sum_{\nu=1}^{K} C_{\alpha\beta}^{(\nu)}(\tau) \,. \tag{2.5}$$

This additive decomposition obviously transfers to their Fourier transforms and thus, also to the final Liouvillian (to second order in the coupling)

$$\mathcal{L} = \mathcal{L}^{(0)} + \sum_{\nu=1}^{K} \mathcal{L}^{(\nu)} \,. \tag{2.6}$$

Here, $\mathcal{L}^{(0)} \rho \hat{=} -\mathrm{i}\,[H_S, \rho]$ describes the action of the system Hamiltonian on the system density matrix ρ and $\mathcal{L}^{(\nu)}$ denotes the Liouvillian resulting only from the ν-th reservoir. The resulting stationary state is in general a non-equilibrium one. Furthermore, the system will in general support a stationary heat current between the two reservoirs.

2.1 Example: Single electron transistor

First we consider a reduced version, the single resonant level. It consists of a single quantum dot that is tunnel-coupled to a continuum

$$H = \epsilon d^\dagger d + \sum_k \left(t_k d^\dagger c_k + t_k^* c_k^\dagger d \right) + \sum_k \epsilon_k c_k^\dagger c_k \,. \tag{2.7}$$

Here, the d and c_k operators obey fermionic anti-commutation relations also between system and bath, such that the above does not comply with the usual tensor product assumption. However,

it can be restored by the Jordan-Wigner transform

$$\begin{aligned} d &= \sigma^- \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{\text{bath modes}} = \sigma_0^- \,, \\ c_1 &= \sigma^z \otimes \underbrace{\sigma^- \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{\text{bath modes}} = \sigma_0^z \sigma_1^- \,, \\ &\vdots \\ c_k &= \sigma^z \otimes \underbrace{\sigma^z \otimes \ldots \otimes \sigma^z \otimes \sigma^- \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{\text{bath modes}} = \sigma_0^z \prod_{\ell=1}^{k-1} \sigma_\ell^z \sigma_k^- \,, \end{aligned} \tag{2.8}$$

which automatically respects the fermionic algebra of all operators. With it, we can rewrite the Hamiltonian as

$$\begin{aligned} H &= \epsilon \, |1\rangle\langle 1| + |1\rangle\langle 0| \otimes \sum_k t_k \tilde{c}_k + |0\rangle\langle 1| \otimes \sum_k t_k^* \tilde{c}_k^\dagger + \sum_k \epsilon_k \tilde{c}_k^\dagger \tilde{c}_k \\ &= \epsilon \tilde{d}^\dagger \tilde{d} + \tilde{d}^\dagger \otimes \sum_k t_k \tilde{c}_k + \tilde{d} \otimes \sum_k t_k^* \tilde{c}_k^\dagger + \sum_k \epsilon_k \tilde{c}_k^\dagger \tilde{c}_k \,, \end{aligned} \tag{2.9}$$

which has the required tensor product form and where the $\tilde{c}_k$ and $\tilde{d}$ operators are new fermions that can be distinguished between system and bath – as if particles would change their nature by traversing the system-reservoir junction. Identifying (we will drop the $\tilde{d}$ and $\tilde{c}_k$ notation and assume that the mapping has been performed tacitly)

$$\begin{aligned} S_1 &= d^\dagger \,, \qquad B_1 = \sum_k t_k c_k \,, \\ S_2 &= d \,, \qquad B_2 = \sum_k t_k^* c_k^\dagger \,, \end{aligned} \tag{2.10}$$

we can determine the non-vanishing correlation functions as

$$C_{12}(\tau) = \frac{1}{2\pi} \int \Gamma(\omega)[1 - f(\omega)] e^{-\mathrm{i}\omega\tau} d\omega \,, \qquad C_{21}(\tau) = \frac{1}{2\pi} \int \Gamma(\omega) f(\omega) e^{+\mathrm{i}\omega\tau} d\omega \,, \tag{2.11}$$

from which we can read off the Fourier transforms of the reservoir correlation functions

$$\gamma_{12}(\tau) = \Gamma(+\omega)[1 - f(+\omega)] \,, \qquad \gamma_{21}(\tau) = \Gamma(-\omega) f(-\omega) \,. \tag{2.12}$$

These determine the dampening coefficients $\gamma_{ab,cd}$ in the BMS Lindblad master equation to

$$\gamma_{01,01} = \gamma_{12}(+\epsilon) = \Gamma(\epsilon)[1 - f(\epsilon)] \,, \qquad \gamma_{10,10} = \gamma_{21}(-\epsilon) = \Gamma(\epsilon) f(\epsilon) \,. \tag{2.13}$$

Inserting this into the master equation and re-expressing it with fermionic system operators we get

$$\begin{aligned} \dot{\rho} = &-\mathrm{i} \left[\epsilon d^\dagger d, \rho\right] + \Gamma(\epsilon)[1 - f(\epsilon)] \left[d\rho d^\dagger - \frac{1}{2} \left\{ d^\dagger d, \rho \right\} \right] \\ &+ \Gamma(\epsilon) f(\epsilon) \left[d\rho d^\dagger - \frac{1}{2} \left\{ d^\dagger d, \rho \right\} \right] . \end{aligned} \tag{2.14}$$

The single-electron transistor (SET) constitutes the simplest possible transport model: It can be understood as a quantum dot that is tunnel-coupled to two reservoirs and thus upgrades the single resonant level to a two-terminal system. From the above discussion of additivity we would get an additive Liouvillian of the form

$$\dot{\rho} = -\mathrm{i}\left[\epsilon d^\dagger d, \rho\right] + \sum_\nu \Gamma_\nu(\epsilon)[1 - f_\nu(\epsilon)]\left[d\rho d^\dagger - \frac{1}{2}\left\{d^\dagger d, \rho\right\}\right] + \Gamma_\nu(\epsilon) f_\nu(\epsilon)\left[d\rho d^\dagger - \frac{1}{2}\left\{d^\dagger d, \rho\right\}\right], \tag{2.15}$$

where $\Gamma_\nu(\omega) = 2\pi \sum_k |t_{k\nu}|^2 \delta(\omega - \epsilon_{k\nu})$ is the spectral coupling density of reservoir $\nu \in \{L, R\}$ and $f_\nu(\omega) = [e^{\beta_\nu(\omega-\mu_\nu)} + 1]^{-1}$ is the Fermi function of reservoir $\nu \in \{L, R\}$. Quite simple observations already tell us that the dot will dominantly be empty when $\epsilon \gg \mu_\nu$ and filled when $\epsilon \ll \mu_\nu$, and transport at steady state will only be possible when e.g. $\mu_L > \epsilon > \mu_R$ (or vice versa).

2.2 Phenomenologic definition of currents

Strictly speaking, a conventional master equation only tells us about the state of the system and not about the changes in the reservoir. For a system that is coupled to a single reservoir, we might from total conservation laws and the dynamics of the system conclude how much energy or how many particles have passed into the reservoir. This is different however for multiple reservoirs, which at non-equilibrium may give rise to steady-state currents. However, the additive decomposition of the Liouville superoperators allows us to phenomenologically identify contributions to the currents from individual reservoirs.

From Eq. (2.6) we can conclude for the energy of the system

$$\frac{d}{dt}\langle E\rangle = \mathrm{Tr}\left\{H_S\dot{\rho}\right\} = -\mathrm{i}\mathrm{Tr}\left\{H_S[H_S, \rho]\right\} + \sum_\nu \mathrm{Tr}\left\{H_S(\mathcal{L}^{(\nu)}\rho)\right\}. \tag{2.16}$$

We immediately see that the first term vanishes under the trace, and that the contributions of the individual reservoirs is additive. This gives rise to the definition of the energy current entering the system from reservoir ν

$$I_E^{(\nu)} = \mathrm{Tr}\left\{H_S(\mathcal{L}^{(\nu)}\rho)\right\} = \mathrm{Tr}\left\{\mathcal{H}_S\mathcal{L}^{(\nu)}\rho\right\}. \tag{2.17}$$

Similarly, we can define a particle current. This only makes sense if the system Hamiltonian conserves the total particle number $[N_S, H_S] = 0$, which leads to

$$\frac{d}{dt}\langle N\rangle = \mathrm{Tr}\left\{N_S\dot{\rho}\right\} = -\mathrm{i}\mathrm{Tr}\left\{N_S[H_S, \rho]\right\} + \sum_\nu \mathrm{Tr}\left\{N_S(\mathcal{L}^{(\nu)}\rho)\right\}. \tag{2.18}$$

Again, the commutator term vanishes and the particle (or matter) current entering the system from reservoir ν becomes

$$I_M^{(\nu)} = \mathrm{Tr}\left\{N_S(\mathcal{L}^{(\nu)}\rho)\right\} = \mathrm{Tr}\left\{\mathcal{N}_S\mathcal{L}^{(\nu)}\rho\right\}. \tag{2.19}$$

We note that in these definitions we have mixed superoperator (calligraphic) and operator notations, which explains why we have put some brackets in the expressions. Let us first consider

the simple case where each Liouvillian $\mathcal{L}^{(\nu)}$ has block structure in the system energy eigenbasis separating populations and diagonals, with the evolution of the diagonals being given by the usual rate equation

$$\dot{\rho}_{aa} = \sum_{\nu} \sum_{b} \gamma^{(\nu)}_{ab,ab} \rho_{bb} - \sum_{\nu} \sum_{b} \gamma^{(\nu)}_{ba,ba} \rho_{aa} \,. \tag{2.20}$$

Representing the density matrix, particle number operator, and Hamiltonian in the time-independent energy eigenbasis as

$$\rho = \sum_{a} \rho_{aa} \left|a\right\rangle\!\left\langle a\right| + \sum_{a \neq b} \rho_{ab} \left|a\right\rangle\!\left\langle b\right| \,, \qquad N_S = \sum_{a} N_a \left|a\right\rangle\!\left\langle a\right| \,, \qquad H_S = \sum_{a} H_a \left|a\right\rangle\!\left\langle a\right| \,, \tag{2.21}$$

we see that

$$I^{(\nu)}_M = \sum_{a} N_a \left[\sum_{b} \gamma^{(\nu)}_{ab,ab} \rho_{bb} - \sum_{b} \gamma^{(\nu)}_{ba,ba} \rho_{aa} \right] = \sum_{ab} (N_a - N_b) \gamma^{(\nu)}_{ab,ab} \rho_{bb} \,. \tag{2.22}$$

At steady state $\rho_{bb} \to \bar{\rho}_{bb}$, this corresponds to the traditional definition of the **matter current for rate equations**, given by the steady state occupation multiplied by the transition rate $\gamma^{(\nu)}_{ab,ab}$ and the particle number difference between the new state a and the old state b. In a completely analogous fashion, we obtain for the energy current entering the system from reservoir ν

$$I^{(\nu)}_E = \sum_{a} E_a \left[\sum_{b} \gamma^{(\nu)}_{ab,ab} \rho_{bb} - \sum_{b} \gamma^{(\nu)}_{ba,ba} \rho_{aa} \right] = \sum_{ab} (E_a - E_b) \gamma^{(\nu)}_{ab,ab} \rho_{bb} \,, \tag{2.23}$$

which is the traditional **energy current for rate equations**. One could alternatively have derived these currents as follows: When the system follows transitions between different energy eigenstates from state b to state a, it gains or looses the energy $E_a - E_b$. The probability for such a process to happen in the time interval Δt triggered by the reservoir ν is just $\Delta t \gamma^{(\nu)}_{ab,ab} \rho_{bb}$. Summing over all initial states b and over all final states a while multiplying the probabilities with the energy difference per time $(E_a - E_b)/\Delta t$ then yields the above phenomenological energy current.

We have defined these currents from the perspective of the system. These definitions just require an additive decomposition of the Liouville superoperator, it does actually not need to be of Lindblad form. But can they really be associated with the corresponding change of energy and particle number in the reservoir? Where does e.g. in case of energy balances the energy contained in the interaction Hamiltonian enter? This requires a more careful analysis to be provided later. Below, we will discuss the phenomenologic thermodynamics arising from these definitions.

2.3 Nonequilibrium thermodynamics

We first phrase the necessary prerequisites. Let us assume that we have a system coupled to many reservoirs and subject to slow driving $H_S \to H_S(t)$. The slow-driving assumption is necessary to ensure that all previous approximations are applicable, such that only the parameters in the dissipators become time-dependent, eventually leading to a master equation of the form

$$\dot{\rho} = -\mathrm{i}[H_S(t), \rho] + \sum_{\nu} \mathcal{L}^{(\nu)}(t) \rho \tag{2.24}$$

for the system density matrix ρ (we often drop the index S for brevity).

Denoting the system energy as $E = \mathrm{Tr}\{H_S(t)\rho(t)\}$, we can state the first law of thermodynamics for the system as a balance equation

$$\begin{aligned}\dot{E} &= \frac{d}{dt}\mathrm{Tr}\{H_S(t)\rho(t)\} \\ &= \mathrm{Tr}\left\{\dot{H}_S\rho_S\right\} + \sum_\nu \mu_\nu \mathrm{Tr}\left\{N_S(\mathcal{L}^{(\nu)}\rho)\right\} + \sum_\nu \mathrm{Tr}\left\{(H_S - \mu_\nu N_S)(\mathcal{L}^{(\nu)}\rho)\right\} . \end{aligned} \tag{2.25}$$

Here, the first term can be interpreted as **mechanical work rate**

$$\dot{W} = \mathrm{Tr}\left\{\dot{H}_S\rho\right\} , \tag{2.26}$$

the second as **chemical work rate** injected by reservoir ν

$$\dot{W}^{(\nu)} = \mu_\nu \mathrm{Tr}\left\{N_S(\mathcal{L}^{(\nu)}\rho)\right\} , \tag{2.27}$$

and the third as a **heat current** entering the system from reservoir ν

$$\dot{Q}^{(\nu)} = \mathrm{Tr}\left\{(H_S - \mu_\nu N_S)(\mathcal{L}^{(\nu)}\rho)\right\} . \tag{2.28}$$

We note that this is not a derivation of the first law. Rather, we have postulated it and used it to classify the individual currents. These definitions remain sensible when H_S is time-dependent.

Furthermore, we assume that also in case of slow time-dependent driving one has that the dissipators $\mathcal{L}^{(\nu)}(t)$ drag towards the time-local Gibbs state

$$\mathcal{L}^{(\nu)}(t)\frac{e^{-\beta_\nu(H_S(t)-\mu_\nu N_S)}}{Z} \equiv \mathcal{L}^{(\nu)}(t)\bar{\rho}^{(\nu)}(t) = 0 . \tag{2.29}$$

In particular, this implies that we can write

$$\ln\bar{\rho}^{(\nu)}(t) = -\beta_\nu(H_S(t) - \mu_\nu N_S) - \ln Z , \tag{2.30}$$

where $\ln Z$ is just a number, such that $\mathrm{Tr}\left\{(\mathcal{L}^{(\nu)}\rho)\ln Z\right\} = 0$. Then, we can show the second law in non-equilibrium as follows

$$\begin{aligned}\dot{S}_\mathrm{i} &= \dot{S} - \sum_\nu \beta_\nu \dot{Q}^{(\nu)} \\ &= -\mathrm{Tr}\{\dot{\rho}\ln\rho\} + \sum_\nu \mathrm{Tr}\left\{[\mathcal{L}^{(\nu)}(t)\rho(t)]\ln\bar{\rho}^{(\nu)}(t)\right\} \\ &= -\sum_\nu \mathrm{Tr}\left\{[\mathcal{L}^{(\nu)}(t)\rho(t)][\ln\rho(t) - \ln\bar{\rho}^{(\nu)}(t)]\right\} , \end{aligned} \tag{2.31}$$

where we have used that $\dot{S} = -\mathrm{Tr}\{\dot{\rho}\ln\rho\} = -\sum_\nu \mathrm{Tr}\left\{(\mathcal{L}^{(\nu)}\rho)\ln\rho\right\}$, since the commutator term does not contribute. With view on Eq. (2.29), we can for each term in the summation use Spohn's inequality to conclude that the **irreversible entropy production rate** is non-negative

$$\dot{S}_\mathrm{i} = \dot{S} - \sum_\nu \beta_\nu \dot{Q}^{(\nu)} = \dot{S} + \sum_\nu \dot{S}^{(\nu)}_\mathrm{res} \geq 0 . \tag{2.32}$$

This denotes the second law in presence of (slow) driving and multiple reservoirs. We stress that we have used very few ingredients to arrive at this result: First, the total Liouvillian is additive in the baths and probability conserving. Second, the stationary state of each Lindblad superoperator is the local thermal equilibrium state, possibly depending on time. Strict positivity of the above entropy production rate in time can also be used to classify a dynamics as Markovian, whereas a dynamics can be considered non-Markovian when the above inequality is violated.

We will now discuss some consequences of this second law.

2.4 Steady-State Dynamics

By steady-state we mean that the system density matrix has reached a stationary value, which will in general be a complicated nonequilibrium steady state. The term steady state also means that for the moment we neglect driving $H_S(t) \to H_S$, and the reservoirs only perform chemical work on the system and exchange heat with it – in other words, only matter and energy currents determine the thermodynamics of the model. Given a finite-dimensional Hilbert space and ergodic dynamics, the von-Neumann entropy of the system will saturate at some point $\dot{S} \to 0$ and the entropy production rate is given by the heat flows

$$\dot{S}_{\mathrm{i}} \to -\sum_{\nu} \beta_\nu \dot{\bar{Q}}^{(\nu)} = -\sum_{\nu} \beta_\nu \left[\bar{I}_E^{(\nu)} - \mu_\nu \bar{I}_M^{(\nu)} \right] \geq 0 \,, \tag{2.33}$$

where $\bar{I}_E^{(\nu)}$ and $\bar{I}_M^{(\nu)}$ are the stationary energy and matter currents entering the system from reservoir ν, respectively. Naturally, we see that the entropy production has to vanish when all the average currents vanish (e.g. at a global equilibrium state). Whereas energy and matter conservation imply equalities among the currents at steady state

$$\sum_{\nu} \bar{I}_M^{(\nu)} = 0 \,, \qquad \sum_{\nu} \bar{I}_E^{(\nu)} = 0 \,, \tag{2.34}$$

the positivity of entropy production imposes a further constraint among the currents, e.g. for a two-terminal system

$$\begin{aligned} \dot{S}_{\mathrm{i}} &= -\beta_L (\bar{I}_E^{(L)} - \mu_L \bar{I}_M^{(L)}) - \beta_R (\bar{I}_E^{(R)} - \mu_R \bar{I}_M^{(R)}) \\ &= (\beta_R - \beta_L) \bar{I}_E + (\mu_L \beta_L - \mu_R \beta_R) \bar{I}_M \geq 0 \,, \end{aligned} \tag{2.35}$$

where we have introduced the currents from left to right $\bar{I}_E = +\bar{I}_E^{(L)} = -\bar{I}_E^{(R)}$ and $\bar{I}_M = +\bar{I}_M^{(L)} = -\bar{I}_M^{(R)}$.

We first discuss the case of equal temperatures $\beta = \beta_L = \beta_R$. The second law implies that

$$(\mu_L - \mu_R) \bar{I}_M \geq 0 \,, \tag{2.36}$$

which is nothing but the trivial statement that the matter current is always directed from a lead with large chemical potential towards the lead with smaller chemical potential.

Next, we consider equal chemical potentials $\mu_L = \mu_R = \mu$ but different temperatures. Then, our setup has to obey

$$(\beta_R - \beta_L)(\bar{I}_E - \mu \bar{I}_M) \geq 0 \,, \tag{2.37}$$

where $\bar{I}_E - \mu \bar{I}_M$ is now the heat leaving the left reservoir and entering the right reservoir. When $\beta_R > \beta_L$ (i.e., the left lead is hotter than the right one $T_L > T_R$), the second law just implies that $\bar{I}_E - \mu \bar{I}_M \geq 0$, i.e., the heat has to flow from left to right. Similarly, it has to revert sign when $\beta_R < \beta_L$. Altogether, this only tells us that in absence of driving, heat always flows from hot to cold – the Clausius statement of the second law of thermodynamics.

An interesting scenario arises when there are both a temperature and a potential gradient present, dragging to different directions. For a two-terminal system the second law then reads at steady state

$$(\beta_R - \beta_L) \bar{I}_E + (\mu_L \beta_L - \mu_R \beta_R) \bar{I}_M \geq 0 \,. \tag{2.38}$$

With this, it is possible to use a temperature gradient to drive a current against a potential bias, i.e., to perform chemical work. In case of e.g. electrons driven against an electric bias, this would be a **thermoelectric generator**. Without loss of generality we assume $\mu_L < \mu_R$ and $\beta_L < \beta_R$ (i.e., the left reservoir is hotter than the right one). The **efficiency** of this generator is then given by the ratio of the generated electric power (or chemical work rate) $P = -\bar{I}_M(\mu_L - \mu_R)$ divided by the heat entering the system from the hot reservoir

$$\begin{aligned}
\eta &= \frac{-\bar{I}_M(\mu_L-\mu_R)}{\bar{I}_E-\mu_L\bar{I}_M} = \frac{-(\beta_R-\beta_L)(\mu_L-\mu_R)\bar{I}_M}{(\beta_R-\beta_L)\bar{I}_E-(\beta_R-\beta_L)\mu_L\bar{I}_M} \\
&= \frac{-(\beta_R-\beta_L)(\mu_L-\mu_R)\bar{I}_M}{(\beta_R-\beta_L)\bar{I}_E+(\mu_L\beta_L-\mu_R\beta_R)\bar{I}_M-(\mu_L\beta_L-\mu_R\beta_R)\bar{I}_M-(\beta_R-\beta_L)\mu_L\bar{I}_M} \\
&\leq \frac{-(\beta_R-\beta_L)(\mu_L-\mu_R)\bar{I}_M}{-(\mu_L\beta_L-\mu_R\beta_R)\bar{I}_M-(\beta_R-\beta_L)\mu_L\bar{I}_M} = \frac{(\beta_R-\beta_L)(\mu_L-\mu_R)}{(\mu_L\beta_L-\mu_R\beta_R)+(\beta_R-\beta_L)\mu_L} \\
&= 1-\frac{\beta_L}{\beta_R} = 1-\frac{T_R}{T_L} = 1-\frac{T_{\rm cold}}{T_{\rm hot}} = \eta_{\rm Carnot}\,. \qquad (2.39)
\end{aligned}$$

The efficiency of such a generator is bounded by **Carnot efficiency**, irrespective of the microscopic details. We note that our scenario is different from the classical Carnot or Otto cycles, since our reservoirs are coupled at all times to the system, but it is interesting to see that the same universal law holds.