Open quantum systems – lecture 9

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Chapter 3

Full Counting Statistics

Previous definitions of currents were based on the phenomenologic identification of the change of a system observable (energy, particle number) with additive contributions from the reservoirs. Sometimes however one is also interested in more information beyond the mean values, i.e., the statistics of single jumps into the reservoir. In Full Counting Statistics (FCS) for example one is interested in the probability distribution $P_n(t)$ denoting the net number of particles $n \in \mathbb{Z}$ transferred to a specific reservoir after time t. This can be generalized to full energy counting statistics, which we will also consider in this chapter.

3.1 Phenomenologic Introduction of counting fields

3.1.1 Single jump type

Suppose that by some method we can identify jump terms between different states in the master equation, i.e., we can separate the total dissipator as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \,, \tag{3.1}$$

where $\mathcal{L}_1$ denotes the jump term and $\mathcal{L}_0$ the jump-free evolution (containing the isolated dynamics of the system or un-monitored jumps). For the rate matrix of the SET for example, we could ask for the total number of jumps over the left barrier. Then, a suitable decomposition would be

$$\mathcal{L}_0 = \begin{pmatrix} -\Gamma_L f_L - \Gamma_R f_R & \Gamma_R(1-f_R) \\ \Gamma_R f_R & -\Gamma_R(1-f_R) - \Gamma_L(1-f_L) \end{pmatrix} \,, \qquad \mathcal{L}_1 = \begin{pmatrix} 0 & \Gamma_L(1-f_L) \\ \Gamma_L f_L & 0 \end{pmatrix} \,. \tag{3.2}$$

We can use the Dyson expansion of the full propagator into a series of jumps

$$\begin{aligned}
\mathcal{P}(t) &= e^{\mathcal{L}_0(t-0)} + \int_0^t e^{\mathcal{L}_0(t-t_1)} \mathcal{L}_1 e^{\mathcal{L}_0(t_1-0)} dt_1 \\
&\quad + \int_0^t dt_2 \int_0^{t_2} dt_1 e^{\mathcal{L}_0(t-t_2)} \mathcal{L}_1 e^{\mathcal{L}_0(t_2-t_1)} \mathcal{L}_1 e^{\mathcal{L}_0(t_1-0)} + \ldots \,, \\
&= \mathcal{P}_0(t-0) + \int_0^t dt_1 \mathcal{P}_0(t-t_1) \mathcal{L}_1 \mathcal{P}_0(t_1-0) \\
&\quad + \int_0^t dt_2 \int_0^{t_2} dt_1 \mathcal{P}_0(t-t_2) \mathcal{L}_1 \mathcal{P}_0(t_2-t_1) \mathcal{L}_1 \mathcal{P}_0(t_1-0) + \ldots \,,
\end{aligned} \tag{3.3}$$

which has the appealing interpretation that we have periods of free evolutions interrupted by single jump events.

This decomposition can alternatively be obtained with the Laplace transform of the propagator

$$\mathcal{P}(z) = \int_0^\infty \mathcal{P}(t) e^{-zt} dt = \sum_{n=0}^\infty \frac{\mathcal{L}^n}{n!} \int_0^\infty t^n e^{-zt} dt = \sum_{n=0}^\infty \frac{\mathcal{L}^n}{n!} \frac{n!}{z^{n+1}} = \frac{1}{z}\left[\mathbf{1} - \frac{\mathcal{L}}{z}\right]^{-1} = [z\mathbf{1} - \mathcal{L}]^{-1} . \tag{3.4}$$

It is simpler to expand the Laplace-transform of the propagator

$$\begin{aligned} \mathcal{P}(z) &= [z\mathbf{1} - \mathcal{L}_0 - \mathcal{L}_1]^{-1} = [(z\mathbf{1} - \mathcal{L}_0)(\mathbf{1} - (z\mathbf{1} - \mathcal{L}_0)^{-1}\mathcal{L}_1)]^{-1} \\ &= (\mathbf{1} - (z\mathbf{1} - \mathcal{L}_0)^{-1}\mathcal{L}_1)^{-1}(z\mathbf{1} - \mathcal{L}_0)^{-1} = (\mathbf{1} - \mathcal{P}_0(z)\mathcal{L}_1)^{-1}\mathcal{P}_0(z) , \end{aligned} \tag{3.5}$$

where $\mathcal{P}_0(z) = [z\mathbf{1} - \mathcal{L}_0]^{-1} = \int_0^\infty e^{\mathcal{L}_0 t} e^{-zt} dt$ is the Laplace transform of the free propagator. Using it, we can expand the full propagator as

$$\mathcal{P}(z) = \sum_{n=0}^\infty [\mathcal{P}_0(z)\mathcal{L}_1]^n \mathcal{P}_0(z) = \mathcal{P}_0(z) + \mathcal{P}_0(z)\mathcal{L}_1\mathcal{P}_0(z) + \mathcal{P}_0(z)\mathcal{L}_1\mathcal{P}_0(z)\mathcal{L}_1\mathcal{P}_0(z) + \ldots . \tag{3.6}$$

This is just the Laplace representation of (3.3).

The benefit of the series expansion (3.3) is that it yields a decomposition where we can readily write down the probabilities for n jump events during time t

$$P_n(t) = \mathrm{Tr}\left\{\int_0^t dt_n \ldots \int_0^{t_2} dt_1 e^{\mathcal{L}_0(t-t_n)}\mathcal{L}_1 \ldots \mathcal{L}_1 e^{\mathcal{L}_0(t_2-t_1)}\mathcal{L}_1 e^{\mathcal{L}_0(t_1-0)}\rho_0\right\} , \tag{3.7}$$

which looks way more convenient in Laplace space

$$P_n(z) = \mathrm{Tr}\left\{[\mathcal{P}_0(z)\mathcal{L}_1]^n \mathcal{P}_0(z)\rho_0\right\} = \int_0^\infty P_n(t) e^{-zt} dt . \tag{3.8}$$

But suppose we are only given the full Lindbladian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. Is there a convenient way to sort out only those contributions that have exactly n jump events, without the need to perform a jump (or Dyson) series expansion?

Taking the identity

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{+\mathrm{i}n\chi} e^{-\mathrm{i}m\chi} d\chi = \delta_{nm} \tag{3.9}$$

into account, it becomes quite obvious that one can infer the statistics of such jumps with following replacement

$$\mathcal{L}_1 \to \mathcal{L}_1 e^{+\mathrm{i}\chi} , \qquad \mathcal{L} \to \mathcal{L}(\chi) = \mathcal{L}_0 + \mathcal{L}_1 e^{+\mathrm{i}\chi} \tag{3.10}$$

in the full propagator $\mathcal{P}(\chi, t) = e^{\mathcal{L}(\chi)t}$. Then, terms with powers $\mathcal{L}_1^n$ go as $e^{+\mathrm{i}n\chi}$, and we can project on them by performing an appropriate integration according to Eq. (3.9).

The new variable χ is conventionally called **counting field**. Then, we can use the orthonormality relation (3.9) to conclude

$$P_n(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{Tr}\left\{\mathcal{P}(\chi, t)\rho_0\right\} e^{-\mathrm{i}n\chi} d\chi , \qquad P_n(z) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{Tr}\left\{\mathcal{P}(\chi, z)\rho_0\right\} e^{-\mathrm{i}n\chi} d\chi . \tag{3.11}$$

The corresponding Moment-generating function is given by the Fourier transform of the probability distribution, and we can infer the definition below.

Def. 1 (Moment- and Cumulant-Generating function)**.** *For a generalized generator $\mathcal{L}(\chi)$, the moment-generating function for an initial state ρ_0 is*

$$M(\chi,t) = \mathrm{Tr}\left\{e^{\mathcal{L}(\chi)t}\rho_0\right\} . \tag{3.12}$$

Once this function is known, all moments can be computed by differentiation with respect to the counting field

$$\left\langle n^k\right\rangle_t = \sum_n n^k P_n(t) = (-\mathrm{i}\partial_\chi)^k M(\chi,t)\Big|_{\chi=0} . \tag{3.13}$$

The cumulant-generating function is given by

$$C(\chi,t) = \ln M(\chi,t)\,, \tag{3.14}$$

and by differentiation with respect to the counting field all cumulants are recovered $\left\langle\left\langle n^k\right\rangle\right\rangle_t = (-\mathrm{i}\partial_\chi)^k C(\chi,t)\Big|_{\chi=0}$.

An easy way to see that moments can be obtained by differentiation with respect to the counting field χ is to invert the FT in Eq. (3.11)

$$M(\chi,t) = \sum_n e^{+\mathrm{i}n\chi} P_n(t)\,. \tag{3.15}$$

This makes it quite obvious that $\left\langle n^k\right\rangle = (-\mathrm{i}\partial_\chi)^k M(\chi,t)\Big|_{\chi\to 0}$. Cumulants and moments are of course related, we just summarize relations for the lowest few cumulants

$$\begin{aligned}
\langle\langle n\rangle\rangle &= \langle n\rangle\ ,\\
\left\langle\left\langle n^2\right\rangle\right\rangle &= \left\langle n^2\right\rangle - \langle n\rangle^2\ ,\\
\left\langle\left\langle n^3\right\rangle\right\rangle &= \left\langle n^3\right\rangle - 3\langle n\rangle\left\langle n^2\right\rangle + 2\langle n\rangle^3\ ,\\
\left\langle\left\langle n^4\right\rangle\right\rangle &= \left\langle n^4\right\rangle - 4\langle n\rangle\left\langle n^3\right\rangle - 3\left\langle n^2\right\rangle^2 + 12\langle n\rangle^2\left\langle n^2\right\rangle - 6\langle n\rangle^4\ .
\end{aligned} \tag{3.16}$$

Obviously, the first two cumulants are just related to the **mean of a distribution** and the **width of a distribution**. For unimodal distributions, the third cumulant (skewness) and the fourth cumulant (kurtosis) describe the shape of the distribution near its maximum. In contrast to moments, higher cumulants are inert when a trivial transformation such as a simple shift is performed on a probability distribution.

As an example, consider the SET, where we would e.g. like to count all jumps over the left barrier – also known as activity of the left barrier. The corresponding tilted Liouvillian would read

$$\mathcal{L}(\chi) = \begin{pmatrix} -[\ldots] & \Gamma_L(1-f_L)e^{+\mathrm{i}\chi} + \Gamma_R(1-f_R) \\ \Gamma_L f_L e^{+\mathrm{i}\chi} + \Gamma_R f_R & -[\ldots] \end{pmatrix} . \tag{3.17}$$

This only conserves the trace when $\chi \to 0$ (and when we generalize this to Lindblad dynamics, the Lindblad form is also only restored when $\chi \to 0$).

3.1.2 Different jump types

So how is it then possible to count different jumps? Regarding the example of the SET, this could mean that one could distinguish jumps into the system over the left barrier and jumps out of the system over the left barrier. We can base this on the already existing expansion. By further splitting the free Liouvillian $\mathcal{L}_0 = \mathcal{L}_{00} + \mathcal{L}_2$ we would obtain the decomposition

$$\mathcal{P}_0(z) = \sum_{m=0}^{\infty} \left[\mathcal{P}_{00}(z)\mathcal{L}_2\right]^m \mathcal{P}_{00}(z)\,, \tag{3.18}$$

which we can insert in Eq. (3.6). The first terms of the resulting expansion would read

$$\begin{aligned}\mathcal{P}(z) = {} & \mathcal{P}_{00}(z) + \mathcal{P}_{00}(z)\mathcal{L}_1\mathcal{P}_{00}(z) + \mathcal{P}_{00}(z)\mathcal{L}_2\mathcal{P}_{00}(z) \\ & + \mathcal{P}_{00}(z)\mathcal{L}_1\mathcal{P}_{00}(z)\mathcal{L}_1\mathcal{P}_{00}(z) + \mathcal{P}_{00}(z)\mathcal{L}_2\mathcal{P}_{00}(z)\mathcal{L}_2\mathcal{P}_{00}(z) \\ & + \mathcal{P}_{00}(z)\mathcal{L}_1\mathcal{P}_{00}(z)\mathcal{L}_2\mathcal{P}_{00}(z) + \mathcal{P}_{00}(z)\mathcal{L}_2\mathcal{P}_{00}(z)\mathcal{L}_1\mathcal{P}_{00}(z) + \ldots\,.\end{aligned} \tag{3.19}$$

However, we see that with the replacement $\mathcal{L}_1 \to \mathcal{L}_1 e^{+\mathrm{i}\chi}$ and $\mathcal{L}_2 \to \mathcal{L}_2 e^{+\mathrm{i}\xi}$ the probability of getting n jumps of type $\mathcal{L}_1$ and m jumps of type $\mathcal{L}_2$ during time $[0, t]$ can be obtained via

$$P_{nm}(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\chi \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\xi \mathrm{Tr}\left\{\exp\left\{\mathcal{L}(\chi,\xi)t\right\}\rho_0\right\} e^{-\mathrm{i}n\chi} e^{-\mathrm{i}m\xi}\,. \tag{3.20}$$

Accordingly, the moment-generating function is then $M(\chi, \xi) = \mathrm{Tr}\left\{\exp\left\{\mathcal{L}(\chi, \xi)t\right\}\rho_0\right\}$.

If for the SET example, we would count jumps corresponding to electrons entering via the left barrier and jumps corresponding to electrons entering via the right barrier, the tilted Liouvillian would become

$$\mathcal{L}(\chi,\xi) = \begin{pmatrix} -[\ldots] & \Gamma_L(1-f_L) + \Gamma_R(1-f_R) \\ \Gamma_L f_L e^{+\mathrm{i}\chi} + \Gamma_R f_R e^{+\mathrm{i}\xi} & -[\ldots] \end{pmatrix}\,. \tag{3.21}$$

3.1.3 Net particle transfers

An important special case arises when we are interested in all trajectories that only lead to a net difference. For example, we may be interested in in the net number of particles leaving the system to a certain reservoir. Then, we could count the outgoing jumps ($\mathcal{L}_+$) and the ingoing jumps ($\mathcal{L}_-$) at first separately. From the resulting distribution $P_{n_{\mathrm{out}},n_{\mathrm{in}}}(t)$ describing the joint distribution of $n_{\mathrm{out}} \in \mathbb{Z}^+$ outgoing and $n_{\mathrm{in}} \in \mathbb{Z}^+$ ingoing jumps during time t, the required information can be reconstructed

$$P_n(t) = \sum_{n_{\mathrm{out}},n_{\mathrm{in}}=0}^{\infty} P_{n_{\mathrm{out}},n_{\mathrm{in}}}(t)\delta_{n_{\mathrm{out}}-n_{\mathrm{in}},n}\,. \tag{3.22}$$

Here, $P_n(t)$ describes the probability of having net $n \in \mathbb{Z}$ particle transfers into the monitored reservoir. To project out the trajectories that only have the net $n = n_{\mathrm{out}} - n_{\mathrm{in}}$ particle exchange, we can simply use the replacement $\mathcal{L}_+ \to \mathcal{L}_+ e^{+\mathrm{i}\chi}$ and $\mathcal{L}_- \to \mathcal{L}_- e^{-\mathrm{i}\chi}$

$$P_n(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\chi \mathrm{Tr}\left\{\mathcal{P}(+\chi,-\chi,t)\rho_0\right\} e^{-\mathrm{i}n\chi}\,. \tag{3.23}$$

This means that we can simply use $\xi = -\chi$ to define the decomposition

$$\mathcal{L}(\chi) = \mathcal{L}_0 + e^{+i\chi}\mathcal{L}_+ + e^{-i\chi}\mathcal{L}_- \tag{3.24}$$

and use the standard definition of the moment-generating function $M(\chi, t) = \mathrm{Tr}\left\{e^{\mathcal{L}(\chi)t}\rho_0\right\}$ and the derived cumulant-generating function. For the SET example, a standard application would be to count electrons entering via the left positively and leaving via the left negatively, which yields

$$\mathcal{L}(\chi) = \begin{pmatrix} -[\ldots] & \Gamma_L(1-f_L)e^{-i\chi} + \Gamma_R(1-f_R) \\ \Gamma_L f_L e^{+i\chi} + \Gamma_R f_R & -[\ldots] \end{pmatrix}. \tag{3.25}$$

3.1.4 Long-term dynamics of cumulants

The clear advantage of the description by cumulants however lies in the fact that the long-term evolution of the cumulant-generating function is usually given by the dominant eigenvalue of the Liouvillian

$$C(\chi, t) \approx \lambda(\chi)t\,, \tag{3.26}$$

where $\lambda(\chi)$ is the (uniqueness assumed) eigenvalue of the Liouvillian that vanishes at zero counting field $\lambda(0) = 0$. For this reason, the dominant eigenvalue is also interpreted as the cumulant-generating function of the stationary current. We recall that the Liouville superoperator is in general non-hermitian and may not have a spectral representation. Nevertheless, we can represent it in Jordan Block form

$$\mathcal{L}(\chi) = Q(\chi)\mathcal{L}_J(\chi)Q^{-1}(\chi)\,, \tag{3.27}$$

where $Q(\chi)$ is a (non-unitary) similarity matrix and $\mathcal{L}_J(\chi)$ contains the eigenvalues of the Liouvillian on its diagonal – distributed in blocks with a size corresponding to the eigenvalue multiplicity. We assume that there exists one stationary state $\bar{\rho}$, i.e., one eigenvalue $\lambda(\chi)$ with $\lambda(0) = 0$ and that all other eigenvalues have a larger negative real part near $\chi = 0$. Then, we use this decomposition in the matrix exponential to estimate its long-term evolution

$$\begin{aligned} M(\chi, t) &= \mathrm{Tr}\left\{e^{\mathcal{L}(\chi)t}\rho_0\right\} = \mathrm{Tr}\left\{e^{Q(\chi)\mathcal{L}_J(\chi)Q^{-1}(\chi)t}\rho_0\right\} = \mathrm{Tr}\left\{Q(\chi)e^{\mathcal{L}_J(\chi)t}Q^{-1}(\chi)\rho_0\right\} \\ &\to \mathrm{Tr}\left\{Q(\chi)\begin{pmatrix} e^{\lambda(\chi)\cdot t} & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}Q^{-1}(\chi)\rho_0\right\} \\ &= e^{\lambda(\chi)\cdot t}\mathrm{Tr}\left\{Q(\chi)\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}Q^{-1}(\chi)\rho_0\right\} = e^{\lambda(\chi)t}c(\chi) \end{aligned} \tag{3.28}$$

with some polynomial $c(\chi)$ depending on the matrix $Q(\chi)$. This implies that the **cumulant-generating function**

$$C(\chi, t) \equiv \ln M(\chi, t) \stackrel{t\to\infty}{\to} \lambda(\chi)t + \ln c(\chi) \approx \lambda(\chi)t \tag{3.29}$$

becomes linear in $\lambda(\chi)$ and t for large times. The cumulants can be conveniently determined once the dominant eigenvalue of the Liouvillian is known.

3.1.5 Dynamics of current and noise

One is not always in the fortunate position to have the dominant eigenvalue of the Liouvillian as an analytic function of χ (as $\mathcal{L}(\chi)$ becomes usually a large matrix). However, for the important quantities current and the noise, one can derive simplified expressions that only involve the time-dependent density matrix.

With the **generalized density matrix**

$$\rho(\chi, t) = e^{\mathcal{L}(\chi)t}\rho_0\,, \tag{3.30}$$

we can adopt the convention that $\rho'(\chi, t) = \partial_\chi \rho(\chi, t)$ and $\dot{\rho}(\chi, t) = \partial_t \rho(\chi, t)$. Then, the time-dependent current becomes

$$\begin{aligned}
I(t) &\equiv \frac{d}{dt}\left\langle n(t)\right\rangle = -\mathrm{i}\partial_\chi \mathrm{Tr}\left\{\frac{d}{dt}\rho(\chi,t)\right\}|_{\chi=0} = -\mathrm{i}\partial_\chi \mathrm{Tr}\left\{\mathcal{L}(\chi)\rho(\chi,t)\right\}|_{\chi=0} \\
&= -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(0,t)\right\} - \mathrm{i}\mathrm{Tr}\left\{\mathcal{L}(0)\rho'(0,t)\right\} \\
&= -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\}\,,
\end{aligned} \tag{3.31}$$

where we have used that $\mathrm{Tr}\left\{\mathcal{L}(0)\hat{O}\right\} = 0$ for any operator $\hat{O}$ due to the trace-conservation obeyed by $\mathcal{L}(0)$. Even simpler, in the long-term limit we may use the stationary state $\mathcal{L}(0)\bar{\rho} = 0$

$$\bar{I} = -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\bar{\rho}\right\}\,. \tag{3.32}$$

The noise becomes

$$\begin{aligned}
S(t) &\equiv \frac{d}{dt}\left(\left\langle n^2(t)\right\rangle - \left\langle n(t)\right\rangle^2\right) = (-\mathrm{i}\partial_\chi)^2 \mathrm{Tr}\left\{\mathcal{L}(\chi)\rho(\chi,t)\right\}|_{\chi=0} - 2I(t)\cdot\left\langle n(t)\right\rangle \\
&= -\mathrm{Tr}\left\{\mathcal{L}''(0)\rho(t)\right\} - 2\mathrm{Tr}\left\{\mathcal{L}'(0)\rho'(0,t)\right\} + 2\left[\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\}\right]\cdot\left[\mathrm{Tr}\left\{\rho'(0,t)\right\}\right] \\
&= -\mathrm{Tr}\left\{\mathcal{L}''(0)\rho(t)\right\} - 2\mathrm{Tr}\left\{\mathcal{L}'(0)\sigma(t)\right\}\,,
\end{aligned} \tag{3.33}$$

where we have defined the auxilary matrix

$$\sigma(t) = \partial_\chi \left.\frac{\rho(\chi,t)}{\mathrm{Tr}\left\{\rho(\chi,t)\right\}}\right|_{\chi=0} = \rho'(0,t) - \rho(t)\mathrm{Tr}\left\{\rho'(0,t)\right\}\,. \tag{3.34}$$

It obeys the differential equation

$$\begin{aligned}
\frac{d}{dt}\sigma(t) &= \partial_\chi \frac{\mathcal{L}(\chi)\rho(\chi,t)}{\mathrm{Tr}\left\{\rho(\chi,t)\right\}}|_{\chi=0} - \partial_\chi \frac{\rho(\chi,t)}{\mathrm{Tr}\left\{\rho(\chi,t)\right\}^2}\mathrm{Tr}\left\{\mathcal{L}(\chi)\rho(\chi,t)\right\}|_{\chi=0} \\
&= \mathcal{L}'(0)\rho(t) + \mathcal{L}(0)\sigma(t) - \rho(t)\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\}\,,
\end{aligned} \tag{3.35}$$

and is subject to the initial condition $\sigma(0) = \mathbf{0}$.

Def. 2 (Current and Noise)**.** *To obtain both time-dependent current $I(t)$ and noise $S(t)$*

$$\begin{aligned}
I(t) &\equiv \frac{d}{dt}\left\langle n\right\rangle_t = -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\}\,, \\
S(t) &\equiv \frac{d}{dt}\left[\left\langle n^2\right\rangle_t - \left\langle n\right\rangle_t^2\right] = -\mathrm{Tr}\left\{\mathcal{L}''(0)\rho(t)\right\} - 2\mathrm{Tr}\left\{\mathcal{L}'(0)\sigma(t)\right\}\,,
\end{aligned} \tag{3.36}$$

one has to solve the coupled (nonlinar) differential equations

$$\begin{aligned}\dot{\rho} &= \mathcal{L}\rho(t)\,,\\ \dot{\sigma} &= \left[\mathcal{L}'(0) - \mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\}\right]\rho(t) + \mathcal{L}(0)\sigma(t)\,,\end{aligned} \tag{3.37}$$

subject to the initial conditions $\rho(0) = \rho_0$ *and* $\sigma(0) = \mathbf{0}$.

Under this evolution, the trace of σ remains conserved, such that $\sigma(t)$ is always a traceless operator. However, when only interested in the long-term limit, we can linearize this by inserting the stationary current. Altogether, at steady state, where $\dot{\rho} = \dot{\sigma} = \mathbf{0}$, this eventually amounts to solving the matrix equation

$$\begin{pmatrix} & \mathcal{L}(0) & \\ - & \mathrm{vec}^{\mathrm{T}}(\mathbf{1}) & - \end{pmatrix} \begin{pmatrix} | \\ \mathrm{vec}(\bar{\rho}) \\ | \end{pmatrix} = \begin{pmatrix} | \\ \mathrm{vec}(\mathbf{0}) \\ | \\ 1 \end{pmatrix} \tag{3.38}$$

first for the steady state $\bar{\rho}$. Then, one solves with the stationary current $\bar{I} = -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\bar{\rho}\right\} = -\mathrm{i}\mathrm{vec}^{\mathrm{T}}(\mathbf{1})\mathcal{L}'(0)\mathrm{vec}(\bar{\rho})$ the equation

$$\begin{pmatrix} & \mathcal{L}(0) & \\ - & \mathrm{vec}^{\mathrm{T}}(\mathbf{1}) & - \end{pmatrix} \begin{pmatrix} | \\ \mathrm{vec}(\bar{\sigma}) \\ | \end{pmatrix} = \begin{pmatrix} | \\ \mathrm{i}\bar{I}\mathrm{vec}(\bar{\rho}) - \mathcal{L}'(0)\mathrm{vec}(\bar{\rho}) \\ | \\ 0 \end{pmatrix} \tag{3.39}$$

for $\bar{\sigma}$. Inserting the results into (3.36) then also yields the stationary noise. Note that in the above two equations, if e.g. $\mathrm{vec}(\rho)$ and $\mathrm{vec}(\sigma)$ are N-dimensional vectors and $\mathcal{L}(0)$ and $\mathcal{L}'(0)$ are $N \times N$ matrices, the vectors on the r.h.s. are $N+1$-dimensional: The purpose of the last row in the matrices on the l.h.s. is to calculate the trace, and the last entry of the vectors on the r.h.s. fixes the trace of $\bar{\rho}$ and $\bar{\sigma}$ to one and zero, respectively.

3.1.6 Example: The harmonic oscillator

Counting particles positively that leave the system and those that enter negatively, the tilted Liouvillian of the harmonic oscillator becomes

$$\mathcal{L}(\chi)\rho \hat{=} -\mathrm{i}\left[\Omega a^\dagger a, \rho\right] + \Gamma(1+n_B)\left[a\rho a^\dagger e^{+\mathrm{i}\chi} - \frac{1}{2}\left\{a^\dagger a, \rho\right\}\right] + \Gamma n_B \left[a^\dagger \rho a e^{-\mathrm{i}\chi} - \frac{1}{2}\left\{a a^\dagger, \rho\right\}\right]\,, \tag{3.40}$$

and we can e.g. compute the time-dependent particle current in a straightforward way

$$I(t) = (-\mathrm{i})\mathrm{Tr}\left\{\mathcal{L}'(0)\rho(t)\right\} = \Gamma(1+n_B)\mathrm{Tr}\left\{a\rho(t)a^\dagger\right\} - \Gamma n_B \mathrm{Tr}\left\{a^\dagger \rho(t) a\right\} = \Gamma\left(\left\langle a^\dagger a\right\rangle_t - n_B\right)\,, \tag{3.41}$$

which vanishes at steady state when $\left\langle a^\dagger a\right\rangle \to n_B$.

3.1.7 Example: The single-electron transistor

We will illustrate these findings with the simple rate equation of the SET with two junctions. For such rate equations, we can naturally interpret the off-diagonal matrix elements as jump terms. Counting, for example the particles entering the system from the left as positive and leaving to the left as negative, we would get the generalized Liouvillian

$$\mathcal{L}(\chi) = \begin{pmatrix} -\Gamma_L f_L & +\Gamma_L(1-f_L)e^{-\mathrm{i}\chi} \\ +\Gamma_L f_L e^{+\mathrm{i}\chi} & -\Gamma_L(1-f_L) \end{pmatrix} + \begin{pmatrix} -\Gamma_R f_R & +\Gamma_R(1-f_R) \\ +\Gamma_R f_R & -\Gamma_R(1-f_R) \end{pmatrix} . \tag{3.42}$$

Application of the trace formula for the current yields the known result

$$\bar{I}_M^{(L)} = -\mathrm{i}\mathrm{Tr}\left\{\mathcal{L}'(0)\bar{\rho}\right\} = (1,1)\begin{pmatrix} 0 & -\Gamma_L(1-f_L) \\ +\Gamma_L f_L & 0 \end{pmatrix}\begin{pmatrix} 1-\bar{f} \\ \bar{f} \end{pmatrix} = \ldots = \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}(f_L-f_R)\,, \tag{3.43}$$

where $\bar{f} = (\Gamma_L f_L + \Gamma_R f_R)/(\Gamma_L+\Gamma_R)$.

The full moment-generating function can be obtained by exponentiating the Liouvillian, but it is simpler to consider its dominant eigenvalue. For simplicity, we will first discuss the **infinite bias regime** $f_L \to 1$ and $f_R \to 0$

$$\mathcal{L}^\infty(\chi) = \begin{pmatrix} -\Gamma_L & \Gamma_R \\ +\Gamma_L e^{+\mathrm{i}\chi} & -\Gamma_R \end{pmatrix} \tag{3.44}$$

Then, we get two eigenvalues

$$\lambda_\pm^\infty(\chi) = \frac{1}{2}\left(-\Gamma_L - \Gamma_R \pm \sqrt{(\Gamma_L-\Gamma_R)^2 + 4e^{+\mathrm{i}\chi}\Gamma_L\Gamma_R}\right)\,, \tag{3.45}$$

and it is visible that $\lambda_+^\infty(0) = 0$, such that $\lambda_{\mathrm{dom}}^\infty(\chi) = \lambda_+^\infty(\chi)$ is the sought-after generating function for the cumulants. In the long-time limit, the first cumulants become

$$\begin{aligned}
\langle\langle n\rangle\rangle &= \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}t\,, \\
\langle\langle n^2\rangle\rangle &= \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\frac{\Gamma_L^2+\Gamma_R^2}{(\Gamma_L+\Gamma_R)^2}t\,, \\
\langle\langle n^3\rangle\rangle &= \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\frac{\Gamma_L^4 - 2\Gamma_L^3\Gamma_R + 6\Gamma_L^2\Gamma_R^2 - 2\Gamma_L\Gamma_R^3 + \Gamma_R^4}{(\Gamma_L+\Gamma_R)^4}t
\end{aligned} \tag{3.46}$$

At finite bias, the statistics can be computed as well. In particular, we can now evaluate the reliability of the thermoelectric generator that converted heat into electric power. For $\mu_L - \mu_R < 0$, we need to transport electrons from left to right to generate positive electric power from heat. We have already calculated the mean current for the SET before. Repeating this now, we can numerically compute the associated probabilities in the long-term limit

$$P_n(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{Tr}\left\{e^{\mathcal{L}(\chi)t - \mathrm{i}n\chi}\bar{\rho}\right\} d\chi \tag{3.47}$$

and compare these with the mean $\langle n\rangle \approx I_M t = -\mathrm{i}\lambda'(0)t$ and noise $\langle\langle n^2\rangle\rangle \approx S_M t = -\lambda''(0)t$ obtained from the long-term cumulants. This is depicted in Fig. 3.1. While for the short times considered, the machine works stochastically (magenta distribution in the inset), this improves for

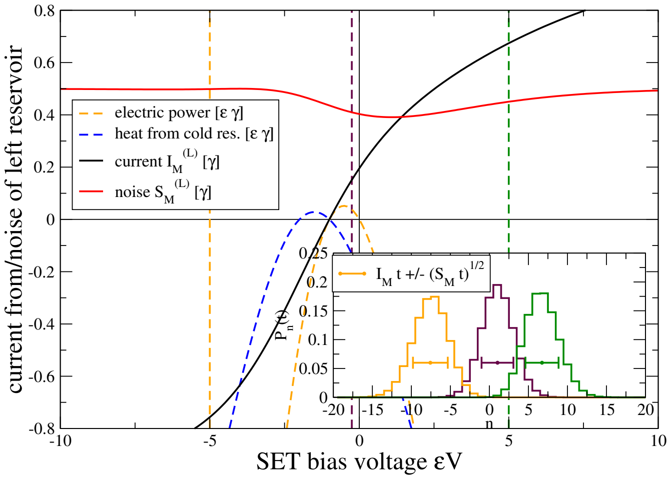

Figure 3.1: Stationary matter current through the SET versus bias voltage. The dashed curves indicate the generated electric power and the heat current entering from the cold (right) reservoir, and there are two regions where the device acts as a heat engine or refrigerator. The inset shows the distribution of the net number of particles leaving the left reservoir after time t at the vertically indicated positions. Parameters: $\Gamma_L = \Gamma_R = 2\gamma$, $\beta_L\epsilon = 0.5$, $\beta_R\epsilon = 1.5$, $\gamma t = 10$.

large times. The mean of the distribution grows linearly in time, and so does the second cumulant. The width however will only grow as $\sqrt{t}$, such that for long times, the heat engine can be considered as reliable.

Alternatively, we can count different things, e.g. the jumps over the right junction, the total number of outgoing or ingoing jumps, the total number of jumps etc. For example, one may be interested in the total number of jumps when the dot is only coupled to a single equilibrium reservoir

$$\mathcal{L}(\chi) = \Gamma \begin{pmatrix} -f & +(1-f)e^{+i\chi} \\ +fe^{+i\chi} & -1+f \end{pmatrix} . \tag{3.48}$$

The dominant eigenvalue is given by

$$\lambda(\chi) = \frac{\Gamma}{2}\left(-1+\sqrt{1-4\left(1-e^{+2i\chi}\right)f(1-f)}\right) . \tag{3.49}$$

From it, we can determine the average value of total jumps for long times

$$\langle n\rangle = 2\Gamma t f(1-f) \le \frac{\Gamma t}{2} . \tag{3.50}$$

One concludes that the average number of jumps vanishes at zero temperature (where either $f=0$ or $f=1$) and becomes maximal at infinite temperature (where $f \to 1/2$).

3.1.8 Example: Absorbtion refrigerator

Revisiting our example of the quantum absorbtion refrigerator, we can generalize the rate matrix of the quantum absorbtion refrigerator to

$$\mathcal{L}_3(\chi) = \begin{pmatrix} -\Gamma_c n_c - \Gamma_h n_h & \Gamma_c(1+n_c)e^{-i\chi} & \Gamma_h(1+n_h) \\ \Gamma_c n_c e^{+i\chi} & -\Gamma_c(1+n_c)-\Gamma_w n_w & \Gamma_w(1+n_w) \\ \Gamma_h n_h & \Gamma_w n_w & -\Gamma_h(1+n_h)-\Gamma_w(1+n_w) \end{pmatrix} . \tag{3.51}$$

This allows to compute the average number of cooling cycle operations, which is positive if on average we are cooling the cold reservoirs and becomes negative when on average we are heating it. The FCS now allows for more detailed statements on the reliability of this engine. Using the

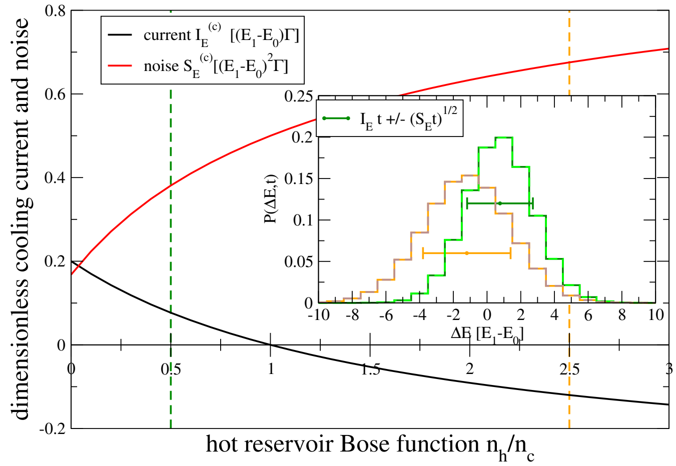

Figure 3.2: Plot of the energy current and noise from the cold reservoir vs. Bose distribution of the hot reservoir (in units of the cold reservoir distribution). The inset (taken at vertical dashed lines of same color) demonstrates that the mean of the distribution is slightly positive in the cooling regime, but a large noise spoils the reliability of the cooler. Solid curves correspond to Eq. (3.51), whereas dashed curves correspond to Eq. (3.56). Parameters: $\Gamma_c = \Gamma_h = \Gamma_w = \Gamma$, $n_w = 1000$, $n_c = 1$, $\Gamma\Delta t = 10$.

trace formula for the current, we reproduce the current for the absorbtion refrigerator up to a factor of $E_1 - E_0$, which is due to the fact that we are not counting energies but absorbtion and emission events. The FCS analysis shows that the device does not work deterministically, see Fig. 3.2.

In general, the dominant eigenvalue of a 3×3 matrix is hard to obtain. However, in the interesting limit where $n_w \to \infty$, we see that the transition between the levels 1 and 2 is much faster than the others, such that we may coarse-grain the equation to reduce its dimension. Introducing

$$\rho_{12} = \rho_{11} + \rho_{22}\,, \tag{3.52}$$

we can write the equation for all populations

$$\frac{d}{dt}\begin{pmatrix}\rho_{00}\\ \rho_{11}\\ \rho_{22}\end{pmatrix} = \begin{pmatrix} -\Gamma_c n_c - \Gamma_h n_h & \Gamma_c(1+n_c)e^{-\mathrm{i}\chi} & \Gamma_h(1+n_h)\\ \Gamma_c n_c e^{+\mathrm{i}\chi} & -\Gamma_c(1+n_c) - \Gamma_w n_w & \Gamma_w(1+n_w)\\ \Gamma_h n_h & \Gamma_w n_w & -\Gamma_h(1+n_h) - \Gamma_w(1+n_w)\end{pmatrix}\begin{pmatrix}\rho_{00}\\ \rho_{11}\\ \rho_{22}\end{pmatrix} \tag{3.53}$$

as a reduced (but non-Markovian equation for the probabilities ρ_{00} and ρ_{12}

$$\begin{aligned}\dot\rho_{00} &= -[\Gamma_c n_c + \Gamma_h n_h]\rho_{00} + \left[\Gamma_c(1+n_c)e^{-\mathrm{i}\chi}\frac{\rho_{11}}{\rho_{12}} + \Gamma_h(1+n_h)\frac{\rho_{22}}{\rho_{12}}\right]\rho_{12}\\ \dot\rho_{12} &= +[\Gamma_c n_c e^{+\mathrm{i}\chi} + \Gamma_h n_h]\rho_{00} - \left[\Gamma_c(1+n_c)\frac{\rho_{11}}{\rho_{12}} + \Gamma_h(1+n_h)\frac{\rho_{22}}{\rho_{12}}\right]\rho_{12}\,.\end{aligned} \tag{3.54}$$

This is non-Markovian, since in we still need to know ρ_{11} and ρ_{22} to compute the rates in the large brackets. However, observing that $\frac{\rho_{11}}{\rho_{12}}$ and $\frac{\rho_{22}}{\rho_{12}}$ are just the conditional probabilities of being in state 1 or 2, provided the system is in the coarse-grained state 12, we can replace these values in the limit $n_w \to \infty$ by their stationary values

$$\lim_{n_w\to\infty}\frac{\rho_{11}}{\rho_{12}} = \lim_{n_w\to\infty}\frac{\rho_{22}}{\rho_{12}} = \frac{1}{2}\,, \tag{3.55}$$

which then leaves us with a reduced Markovian equation for just two variables

$$\frac{d}{dt}\begin{pmatrix}\rho_{00}\\ \rho_{12}\end{pmatrix} = \begin{pmatrix} -\Gamma_c n_c - \Gamma_h n_h & \frac{\Gamma_c}{2}(1+n_c)e^{-\mathrm{i}\chi} + \frac{\Gamma_h}{2}(1+n_h)\\ +\Gamma_c n_c e^{+\mathrm{i}\chi} + \Gamma_h n_h & -\frac{\Gamma_c}{2}(1+n_c) - \frac{\Gamma_h}{2}(1+n_h)\end{pmatrix}\begin{pmatrix}\rho_{00}\\ \rho_{12}\end{pmatrix}\,. \tag{3.56}$$

For this, the dominant eigenvalue can be computed analytically, and the agreement of the resulting FCS with that of the full model is excellent for large n_w (compare inset of Fig. 3.2).