Open quantum systems – lecture 10

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3.2 Microscopic derivation of counting fields

Sometimes, we are interested not only in the number of particles but also e.g. in the energy transferred into the reservoir. Alternatively, one could be interested in other observables of the reservoir, where at the level of the Hamiltonian it is not immediately apparent how these reservoir observables are changed by individual terms. Therefore, we also consider another microscopic way of deriving generalized master equations here.

3.2.1 Two-point measurement scheme

At this point, we only assume that the reservoir observable of interest $\hat{O}$ commutes with the reservoir Hamiltonian $[\hat{O}, H_B] = 0$. The observable in the reservoir can already initially take infinite values – after all, a reservoir can contain an infinite amount of particles. To say by how much the observable has changed during some time interval t, one formally introduces a two-point measurement scheme [1]. The first measurement at time 0 defines the initial value of the observable, and the second measurement at time t its final value. The difference then tells us by how much the observable has changed in between (e.g., how many particles have tunnelled into the reservoir).

We will employ the spectral decomposition of the observable (mostly, one is concerned with observables such as the Hamiltonian $\hat{H}_B$ or the particle number operator $\hat{N}_B$ of the reservoir)

$$\hat{O} = \sum_\ell O_\ell \left|\ell\right\rangle\left\langle\ell\right| . \tag{3.1}$$

Upon the specific outcome ℓ, the initial measurement projects the bath density matrix to

$$\bar{\rho}_B \xrightarrow{\ell} \frac{\left|\ell\right\rangle\left\langle\ell\right| \bar{\rho}_B \left|\ell\right\rangle\left\langle\ell\right|}{P_\ell} = \frac{\bar{\rho}_B^{(\ell)}}{P_\ell} , \tag{3.2}$$

where $P_\ell = \mathrm{Tr}\left\{\left|\ell\right\rangle\left\langle\ell\right| \bar{\rho}_B\right\}$ denotes the probability for the outcome ℓ. Since we only measure a reservoir observable and initially, system and reservoir are assumed to be in a product state $\rho_{\rm tot}(0) = \rho_S(0) \otimes \bar{\rho}_B$, this does not affect the system density matrix. The initial value O_ℓ is now our reference point with respect to which we define the change. Since we do not only want a generating function specific to a certain initial value, we perform a weighted average over all outcomes to define the moment-generating function

$$M(\chi, t) = \sum_\ell \mathrm{Tr}\left\{e^{\mathrm{i}\chi(\hat{O}-O_\ell)} \boldsymbol{U}(t) \rho_S^0 \otimes \bar{\rho}_B^{(\ell)} \boldsymbol{U}^\dagger(t)\right\} , \tag{3.3}$$

where we see that the probability P_ℓ has cancelled due to the weighted average. We note that this equation has been written down in the interaction picture, where due to our assumption $[H_B, \hat{O}] = 0$, the reservoir observable did not pick up a time dependence. Clearly, computing derivatives with respect to χ pulls down powers of $(\hat{O} - O_\ell)$ in the usual way, such that the above moment-generating function generates moments of the distribution of observable changes with respect to the initial measurement.

We now rewrite the moment-generating function as

$$
\begin{aligned}
M(\chi,t) &= \sum_\ell \mathrm{Tr}\left\{e^{+\mathrm{i}(\hat{O}-O_\ell)\chi}\boldsymbol{U}(t)\rho_S^0\otimes\rho_B^{(\ell)}\boldsymbol{U}^\dagger(t)\right\}\\
&= \sum_\ell \mathrm{Tr}\left\{e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{U}(t)e^{-\mathrm{i}O_\ell\frac{\chi}{2}}\rho_S^0\otimes\rho_B^{(\ell)}e^{-\mathrm{i}O_\ell\frac{\chi}{2}}\boldsymbol{U}^\dagger(t)e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\right\}\\
&= \sum_\ell \mathrm{Tr}\left\{e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{U}(t)e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}\rho_S^0\otimes\rho_B^{(\ell)}e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{U}^\dagger(t)e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\right\}\\
&= \mathrm{Tr}\left\{\boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\rho_0\otimes\left(\sum_\ell\bar{\rho}_B^{(\ell)}\right)\boldsymbol{U}^\dagger_{-\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\right\} = \mathrm{Tr}\left\{\boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\rho_0\otimes\bar{\bar{\rho}}_B\boldsymbol{U}^\dagger_{-\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\right\}\,,\\
\bar{\bar{\rho}}_B &\equiv \sum_\ell \left|\ell\right\rangle\left\langle\ell\right|\bar{\rho}_B\left|\ell\right\rangle\left\langle\ell\right|\,,\qquad \boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\equiv e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{U}(t)e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}\,.
\end{aligned}
\tag{3.4}
$$

Here, we have used that O_ℓ is just a number (first line) and also that $e^{-\mathrm{i}O_\ell\chi/2}\bar{\rho}_B^{(\ell)}e^{-\mathrm{i}O_\ell\chi/2} = e^{-\mathrm{i}\hat{O}\chi/2}\bar{\rho}_B^{(\ell)}e^{-\mathrm{i}\hat{O}\chi/2}$ by construction, cf. Eq. (3.2). The quantity $\bar{\rho}_B$ denotes the averaged initial density matrix after the projection, depending on measurement and initial state, this may or may not have any effect on the statistics. In the cases where $\bar{\rho}_B$ is already diagonal in the system energy eigenbasis and we measure an observable like energy, we simply get that $\bar{\bar{\rho}}_B = \bar{\rho}_B$. One can easily see that the generalized time evolution operator $\boldsymbol{U}_{\boldsymbol{\chi/2}}(t)$ (which respects the same initial condition as the normal time evolution operator) obeys

$$
\begin{aligned}
&\frac{d}{dt}\boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t) = -\mathrm{i}\boldsymbol{H_I}\left(\frac{\chi}{2},t\right)\boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\,,\qquad \frac{d}{dt}\boldsymbol{U}^\dagger_{-\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t) = +\mathrm{i}\boldsymbol{U}^\dagger_{-\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)\boldsymbol{H_I}\left(-\frac{\chi}{2},t\right)\,,\\
&\boldsymbol{H_I}\left(\frac{\chi}{2},t\right)\equiv e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{H_I}(t)e^{-\mathrm{i}\hat{O}\frac{\chi}{2}} = \sum_\alpha \boldsymbol{S_\alpha}(t)\otimes e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{B_\alpha}(t)e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}
\end{aligned}
\tag{3.5}
$$

with a generalized **interaction Hamiltonian with counting field**. To write the moment-generating function approximately as $M(\chi,t)\approx\mathrm{Tr}\left\{\rho(\chi,t)\right\}$ with generalized system density matrix $\rho(\chi,t)$, we now follow – assuming $[\hat{O},\bar{\bar{\rho}}_B]=0$ – the same steps used in the derivation of standard master equations, only using the generalized time evolution operator. Collecting the terms in second order, we see that terms coming to the left of the density matrix are generated by $\boldsymbol{U}_{+\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)$, which lead to correlation functions like

$$
\mathrm{Tr_B}\left\{e^{+\mathrm{i}\hat{O}\chi/2}\boldsymbol{B_\alpha}(t_1)e^{-\mathrm{i}\hat{O}\chi/2}e^{+\mathrm{i}\hat{O}\chi/2}\boldsymbol{B_\beta}(t_2)e^{-\mathrm{i}\hat{O}\chi/2}\bar{\bar{\rho}}_B\right\} = C^0_{\alpha\beta}(t_1,t_2) = C_{\alpha\beta}(t_1,t_2)\,,
\tag{3.6}
$$

such that the dependence on the counting field χ drops out and the usual correlation function applies. Analogously, terms coming to the right of the density matrix are generated by $\boldsymbol{U}^\dagger_{-\frac{\boldsymbol{\chi}}{\boldsymbol{2}}}(t)$, and also here the dependence on the counting field will drop out

$$
\mathrm{Tr_B}\left\{\bar{\bar{\rho}}_B e^{-\mathrm{i}\hat{O}\chi/2}\boldsymbol{B_\alpha}(t_1)e^{+\mathrm{i}\hat{O}\chi/2}e^{-\mathrm{i}\hat{O}\chi/2}\boldsymbol{B_\beta}(t_2)e^{+\mathrm{i}\hat{O}\chi/2}\right\} = C^0_{\alpha\beta}(t_1,t_2) = C_{\alpha\beta}(t_1,t_2)\,.
\tag{3.7}
$$

In contrast, in the terms where the reservoir density matrix is sandwiched by two coupling operators, the operator on the left is transformed with a different sign of the counting field than the operator on the right. In these terms, a contribution can remain, and we therefore define the generalized correlation function below.

Def. 1 (Generalized Correlation Function)**.** *The generalized reservoir correlation function is defined as*

$$C^{\chi}_{\alpha\beta}(t_1,t_2) = \mathrm{Tr}\left\{e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{B_\alpha}(t_1)e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}e^{+\mathrm{i}\hat{O}\frac{\chi}{2}}\boldsymbol{B_\beta}(t_2)e^{-\mathrm{i}\hat{O}\frac{\chi}{2}}\bar{\bar{\rho}}_B\right\}\,. \tag{3.8}$$

If in addition $[H_B,\bar{\bar{\rho}}_B]=0$*, we can define the single-argument functions*

$$C^{\chi}_{\alpha\beta}(\tau) = \mathrm{Tr}\left\{e^{-\mathrm{i}\hat{O}\chi}\boldsymbol{B_\alpha}(\tau)e^{+\mathrm{i}\hat{O}\chi}B_\beta\bar{\bar{\rho}}_B\right\}\,,\qquad \gamma^{\chi}_{\alpha\beta}(\omega) = \int C^{\chi}_{\alpha\beta}(\tau)e^{+\mathrm{i}\omega\tau}d\tau\,. \tag{3.9}$$

So in effect, all derivations go through as before, the difference is that the generalized correlation functions have to be used in terms where the system density matrix is sandwiched. Furthermore, if $\hat{O}=H_B$ is the reservoir energy, we find simply that $C^{\chi}_{\alpha\beta}(\tau)=C_{\alpha\beta}(\tau-\chi)$ and $\gamma^{\chi}_{\alpha\beta}(\omega)=e^{+\mathrm{i}\omega\chi}\gamma_{\alpha\beta}(\omega)$.

This definition can be used to complete the master equation to the counting-field dependent case.

Def. 2 (Generalized BMS master equation)**.** *For an interaction Hamiltonian* $H_I=\sum_\alpha S_\alpha\otimes B_\alpha$ *with reservoir observable* $\hat{O}$*, the BMS master equation is generalized to*

$$\begin{aligned}
\dot{\rho}_S &= -\mathrm{i}\left[H_S+\sum_{ab}\sigma_{ab}L_{ab},\rho_S\right]+\sum_{abcd}\left[\gamma^{\chi}_{ab,cd}L_{ab}\rho_S L^\dagger_{cd}-\frac{\gamma^0_{ab,cd}}{2}\left\{L^\dagger_{cd}L_{ab},\rho_S\right\}\right]\,,\\
\sigma_{ab} &= \frac{\delta_{E_a,E_b}}{2\mathrm{i}}\sum_{\alpha\bar{\alpha}}\sigma_{\alpha\bar{\alpha}}(E_b-E_c)\langle c|\,S_{\bar{\alpha}}\,|b\rangle\langle c|\,S^\dagger_\alpha\,|a\rangle^*\,,\\
\gamma^{\chi}_{ab,cd} &= \delta_{E_b-E_a,E_d-E_c}\sum_{\alpha\bar{\alpha}}\gamma^{\chi}_{\alpha\bar{\alpha}}(E_b-E_a)\langle a|\,S_{\bar{\alpha}}\,|b\rangle\langle c|\,S^\dagger_\alpha\,|d\rangle^*\,.
\end{aligned} \tag{3.10}$$

We see that the counting-field dependence only affects the terms with the density matrix in the middle, which we would phenomenologically have identified as jump terms. The solution to this master equation defines a generalized density matrix for the system $\rho_S(\chi,t)$, and the moment-generating function is then obtained via $M(\chi,t)=\mathrm{Tr}\left\{\rho_S(\chi,t)\right\}$ as usual. If we choose $\hat{O}=H_B$ or $\hat{O}=N_B$ to count energy or particle number of the reservoir, the convention is such that contributions transferred into the reservoir count positive.

3.2.2 Example: transient SRL energy current

Let us consider the energy current entering the single resonant level (SRL), i.e., a single quantum dot coupled to a reservoir. The Hamiltonian reads (we implicitly use the mapping to tensor products)

$$H=\epsilon d^\dagger d + d\otimes\sum_k t_k c_k^\dagger + d^\dagger\otimes\sum_k t_k^* c_k + \sum_k \epsilon_k c_k^\dagger c_k\,. \tag{3.11}$$

We define $B_1=\sum_k t_k c_k^\dagger$, $B_2=B_1^\dagger$, $S_1=d$, and $S_2=d^\dagger$. If we are interested in the energy entering the reservoir $\hat{O}=H_B$, the observable obviously commutes with the reservoir density matrix, when this is held at a thermal state. Then, since the bath density matrix is already diagonal in the measurement basis, we also have $\bar{\bar{\rho}}_B = \bar{\rho}_B$. The generalized correlation functions then become

$$\begin{aligned} C_{12}^{\chi}(\tau) &= \frac{1}{2\pi}\int \Gamma(\omega) f(\omega) e^{-\mathrm{i}\omega\chi} e^{+\mathrm{i}\omega\tau} d\omega = \frac{1}{2\pi}\int \Gamma(-\omega) f(-\omega) e^{+\mathrm{i}\omega\chi} e^{-\mathrm{i}\omega\tau}\,, \\ C_{21}^{\chi}(\tau) &= \frac{1}{2\pi}\int \Gamma(\omega)[1-f(\omega)] e^{+\mathrm{i}\omega\chi} e^{-\mathrm{i}\omega\tau}\,. \end{aligned} \tag{3.12}$$

From this, we can read off the Fourier transforms of the correlation functions

$$\gamma_{12}^{\chi}(\omega) = \Gamma(-\omega) f(-\omega) e^{+\mathrm{i}\omega\chi}\,, \qquad \gamma_{21}^{\chi}(\omega) = \Gamma(+\omega)[1-f(+\omega)] e^{+\mathrm{i}\omega\chi}\,. \tag{3.13}$$

When we want to evaluate the BMS rate equation, we get for the transition rates

$$\gamma_{ab,ab}^{\chi} = \sum_{\alpha\beta} \gamma_{\alpha\beta}^{\chi}(E_b - E_a) \langle a| S_\beta |b\rangle \langle a| S_\alpha^\dagger |b\rangle^*\,, \tag{3.14}$$

which in our case become dependent on the counting field

$$\gamma_{01,01}^{\chi} = \gamma_{21}^{\chi}(+\epsilon) = \Gamma(\epsilon)[1-f(\epsilon)] e^{+\mathrm{i}\epsilon\chi}\,, \qquad \gamma_{10,10}^{\chi} = \gamma_{12}^{\chi}(-\epsilon) = \Gamma(\epsilon) f(\epsilon) e^{-\mathrm{i}\epsilon\chi}\,, \tag{3.15}$$

and our generalized rate matrix becomes

$$\mathcal{L}(\chi) = \Gamma(\epsilon) \begin{pmatrix} -f(\epsilon) & +[1-f(\epsilon)] e^{+\mathrm{i}\epsilon\chi} \\ +f(\epsilon) e^{-\mathrm{i}\epsilon\chi} & -[1-f(\epsilon)] \end{pmatrix}\,. \tag{3.16}$$

These are precisely the differences we would have guessed from a rate equation representation. The sign convention here has been chosen such that currents count positively when they enter the reservoir. If we would count the number of particles instead, we just need to replace $\epsilon \to 1$ in the above equation. However, for many examples, a full microscopic derivation is actually required to obtain a consistent treatment, see below.

3.2.3 Example: SET monitored by a quantum point contact

The microscopic Hamiltonian for the SET was

$$H_{SET} = \epsilon d^\dagger d + \sum_{k\nu} \left[t_{k\nu} d c_{k\nu}^\dagger + t_{k\nu}^* c_{k\nu} d^\dagger \right] + \sum_{k\nu} \epsilon_{k\nu} c_{k\nu}^\dagger c_{k\nu}\,. \tag{3.17}$$

It would be straightforward to derive the exchange of energy and particles with the two leads $\nu \in \{L, R\}$ of the SET as demonstrated above for the SRL. However, we consider the SET additionally coupled to a quantum point contact (QPC)

$$H_{QPC} = \sum_{\nu} \sum_{k} \varepsilon_{k\nu} \gamma_{k\nu}^\dagger \gamma_{k\nu} \tag{3.18}$$

as depicted in Fig. 3.1. The tunneling through the QPC circuit is modified by the presence of electrons on the SET

$$H_I = \underbrace{\left(1 - \alpha d^\dagger d\right)}_{S} \otimes \underbrace{\sum_{kk'} \left[t_{kk'} \gamma_{kL} \gamma_{k'R}^\dagger + t_{kk'}^* \gamma_{k'R} \gamma_{kL}^\dagger \right]}_{B}\,. \tag{3.19}$$

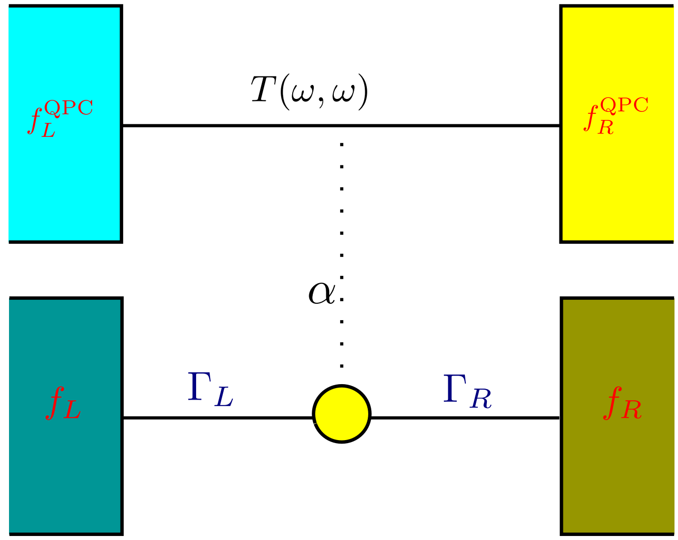

Figure 3.1: Sketch of an SET (bottom) with two leads that is additionally coupled to a QPC (top) consisting of two directly coupled leads. The tunneling through the QPC is modified by the presence of an electron on the SET dot in a way that does not transfer energy nor particles from the SET dot (dotted line). The microscopic derivation of counting fields allows to track the particles transferred through the QPC with a generalized master equation for the SET dot alone.

Here, the case $\alpha = 0$ just represents the bare tunneling through the QPC with $t_{kk'}$ denoting tunneling amplitudes and $\gamma_{k\nu}$ annihilating electrons of mode k in the QPC lead ν. The parameter $0 < \alpha < 1$ models a modification of the QPC tunneling in presence of an electron on the SET dot. We note that this QPC reservoir denotes a nonequilibrium reservoir and thereby does not fit the standard setups considered so far.

For brevity we also only consider the coupling operators to the QPC (we have already discussed the SET) and as reservoir observable, we are interested in the number of electrons transferred to the right QPC reservoir

$$\hat{O} = \sum_k \gamma_{kR}^\dagger \gamma_{kR} \,, \tag{3.20}$$

provided the QPC reservoirs are at local thermal equilibrium states

$$\bar{\rho}_B = \frac{e^{-\beta_L \sum_k (\varepsilon_{kL} - \mu_L) \gamma_{kL}^\dagger \gamma_{kL}}}{Z_L} \frac{e^{-\beta_R \sum_k (\varepsilon_{kR} - \mu_R) \gamma_{kR}^\dagger \gamma_{kR}}}{Z_R} = \bar{\bar{\rho}}_B \,. \tag{3.21}$$

Since $[H_{QPC}, \bar{\rho}_B] = 0$, the generalized correlation function becomes

$$\begin{aligned}
C^\xi(\tau) &= \mathrm{Tr}\left\{e^{-\mathrm{i}\hat{O}\xi} e^{+\mathrm{i}H_{QPC}\tau} B e^{-\mathrm{i}H_{QPC}\tau} e^{+\mathrm{i}\hat{O}\xi} B\bar{\rho}_B\right\} \\
&= \sum_{kk'qq'} \mathrm{Tr}\Big\{ \left[t_{kk'}\gamma_{kL}\gamma_{k'R}^\dagger e^{+\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau} e^{-\mathrm{i}\xi} + t_{kk'}^*\gamma_{k'R}\gamma_{kL}^\dagger e^{-\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau} e^{+\mathrm{i}\xi}\right] \times \\
&\qquad \times \left[t_{qq'}\gamma_{qL}\gamma_{q'R}^\dagger + t_{qq'}^*\gamma_{q'R}\gamma_{qL}^\dagger\right] \bar{\rho}_B\Big\} \\
&= \sum_{kk'qq'} t_{kk'}t_{qq'}^* e^{+\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau} e^{-\mathrm{i}\xi} \mathrm{Tr}\left\{\gamma_{kL}\gamma_{k'R}^\dagger\gamma_{q'R}\gamma_{qL}^\dagger\bar{\rho}_B\right\} \\
&\qquad + \sum_{kk'qq'} t_{kk'}^* t_{qq'} e^{-\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau} e^{+\mathrm{i}\xi} \mathrm{Tr}\left\{\gamma_{k'R}\gamma_{kL}^\dagger\gamma_{qL}\gamma_{q'R}^\dagger\bar{\rho}_B\right\} \\
&= e^{-\mathrm{i}\xi}\sum_{kk'} |t_{kk'}|^2 e^{+\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau}[1-f_L(\varepsilon_{kL})]f_R(\varepsilon_{k'R}) \\
&\qquad + e^{+\mathrm{i}\xi}\sum_{kk'} |t_{kk'}|^2 e^{-\mathrm{i}(\varepsilon_{k'R}-\varepsilon_{kL})\tau} f_L(\varepsilon_{kL})[1-f_R(\varepsilon_{k'R})] \\
&= e^{-\mathrm{i}\xi}\frac{1}{(2\pi)^2}\int d\omega \int d\omega' T(\omega,\omega') e^{+\mathrm{i}(\omega'-\omega)\tau}[1-f_L(\omega)]f_R(\omega') \\
&\qquad + e^{+\mathrm{i}\xi}\frac{1}{(2\pi)^2}\int d\omega \int d\omega' T(\omega,\omega') e^{-\mathrm{i}(\omega'-\omega)\tau} f_L(\omega)[1-f_R(\omega')]\,,
\end{aligned} \tag{3.22}$$

where we have introduced the dimensionless **transmission**

$$T(\omega,\omega') = (2\pi)^2 \sum_{kk'} |t_{kk'}|^2 \delta(\omega-\varepsilon_{kL})\delta(\omega'-\varepsilon_{k'R})\,. \tag{3.23}$$

The FT of the correlation function accordingly can be expressed by a convolution integral

$$\begin{aligned}
\gamma^\xi(\Omega) &= \int C^\xi(\tau) e^{+\mathrm{i}\Omega\tau} d\tau \\
&= \frac{e^{-\mathrm{i}\xi}}{2\pi}\int d\omega T(\omega,\omega-\Omega)[1-f_L(\omega)]f_R(\omega-\Omega) + \frac{e^{+\mathrm{i}\xi}}{2\pi}\int d\omega T(\omega,\omega+\Omega) f_L(\omega)[1-f_R(\omega+\Omega)]\,.
\end{aligned} \tag{3.24}$$

These terms have the appealing interpretation of an electron transfer from right to left (first) or from left to right (second term) while the energy Ω is absorbed from the system, and we see that the sign of the counting field matches this interpretation.

Now, to compute the transition rates (for the SET quantum dot we do not have coherences), we have to evaluate

$$\gamma_{ab,ab}^\xi = \gamma^\xi(E_b - E_a)\left|\langle a|\left(\mathbf{1} - \alpha d^\dagger d\right)|b\rangle\right|^2\,. \tag{3.25}$$

Since the coupling operator is just diagonal, the only allowed transision rates are

$$\gamma_{00,00}^\xi = \gamma^\xi(0)\,, \qquad \gamma_{11,11}^\xi = \gamma^\xi(0)|1-\alpha|^2\,, \tag{3.26}$$

Eventually, this leads to the QPC-induced rate matrix

$$\begin{aligned}
\mathcal{L}_{\mathrm{QPC}}(\xi) &= \begin{pmatrix} 1 & 0 \\ 0 & |1-\alpha|^2 \end{pmatrix} f_{\mathrm{QPC}}(\xi)\,, \\
f_{\mathrm{QPC}}(\xi) &= \left[\frac{e^{-\mathrm{i}\xi}-1}{2\pi}\int T(\omega,\omega)[1-f_L(\omega)]f_R(\omega) + \frac{e^{+\mathrm{i}\xi}-1}{2\pi}\int T(\omega,\omega) f_L(\omega)[1-f_R(\omega)]\right]\,,
\end{aligned} \tag{3.27}$$

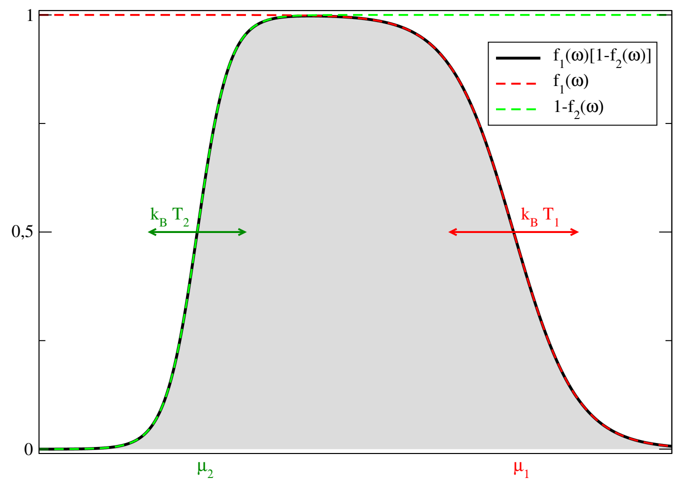

Figure 3.2: Integrand in Eq. (3.29). At zero temperature at both contacts, we obtain a product of two step functions and the area under the curve is given by the difference $\mu_1 - \mu_2$ as soon as $\mu_1 > \mu_2$ (and zero otherwise).

which has to be added to the dissipators of the SET leads. We note that $\mathcal{L}_{\rm QPC}(0) = \mathbf{0}$, such that in ignorance of the QPC results the SET dynamics is unchanged. The function $f_{\rm QPC}(\xi)$ denotes the cumulant-generating function of the unperturbed (low-transparency) QPC, and that $T(\omega, \omega + \Omega)$ is only evaluated at $\Omega = 0$ reflects the fact that the SET and the QPC do not exchange energy, this would be different if e.g. the QPC would monitor a double quantum dot. The fact that this time, the counting fields occur on the diagonal is related to the fact that the QPC charge transfers are not associated to jumps of the SET dot itself. If we assume that the SET dot is empty all the time, the associated QPC current is just determined by the free generating function of the QPC

$$I_0 = (-\mathrm{i}\partial_\xi) f_{\rm QPC}(\xi)|_{\xi=0} = \frac{1}{2\pi} \int T(\omega, \omega)[f_L(\omega) - f_R(\omega)] d\omega \,, \tag{3.28}$$

which is also known as **Landauer formula**. The isolated QPC can also be treated non-perturbatively, such that its full generating function (i.e., nonperturbative $t_{kk'}$) can be derived (see e.g. [2]), such that we have only derived a small $T(\omega, \omega)$ approximation to the full generating function. For a filled SET dot all the time, one simply gets $I_1 = |1 - \alpha|^2 I_0$.

If we assume that $T(\omega, \omega') = T$ is a flat function (at least in the region where the product of the Fermi functions is finite), these integrals can actually be solved analytically: The structure of the Fermi functions demonstrates that the shift Ω can be included in the chemical potentials. Therefore, we consider integrals of the type

$$I = \int f_1(\omega) \left[1 - f_2(\omega)\right] d\omega \,. \tag{3.29}$$

At zero temperature, these should behave as $I \approx (\mu_1 - \mu_2)\Theta(\mu_1 - \mu_2)$, where $\Theta(x)$ denotes the Heaviside-Θ function, which follows from the structure of the integrand, see Fig. 3.2. For finite temperatures, the value of the integral can also be calculated, for simplicity we constrain ourselves to the (experimentally relevant) case of equal temperatures ($\beta_1 = \beta_2 = \beta$), for which we obtain

$$\begin{aligned} I &= \int \frac{1}{\left(e^{\beta(\mu_2 - \omega)} + 1\right)\left(e^{-\beta(\mu_1 - \omega)} + 1\right)} d\omega \\ &= \lim_{\delta \to \infty} \int \frac{1}{\left(e^{\beta(\mu_2 - \omega)} + 1\right)\left(e^{-\beta(\mu_1 - \omega)} + 1\right)} \frac{\delta^2}{\delta^2 + \omega^2} d\omega \,, \end{aligned} \tag{3.30}$$

where we have introduced the Lorentzian-shaped regulator to enforce convergence. By identifying

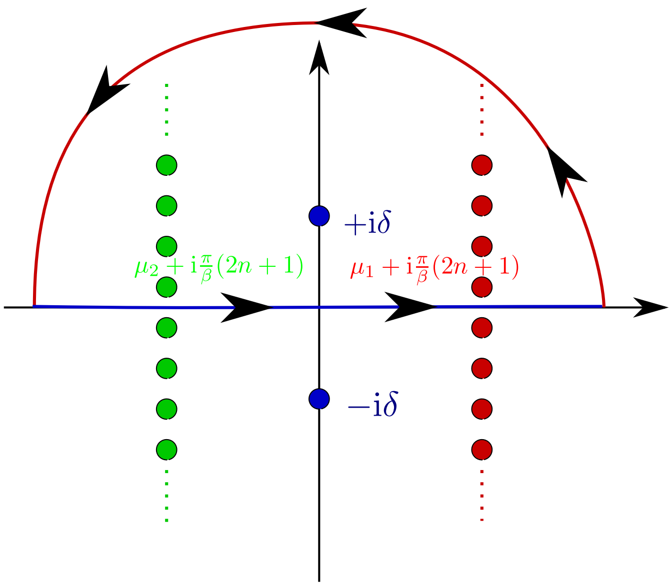

Figure 3.3: Poles and integration contour for Eq. (3.29) in the complex plane. The integral along the real axis (blue line) closed by an arc (red curve) in the upper complex plane, along which (due to the regulator) the integrand vanishes sufficiently fast.

the poles of the integrand

$$\begin{aligned}\omega_\pm^* &= \pm\mathrm{i}\delta\,,\\ \omega_{1,n}^* &= \mu_1 + \mathrm{i}\frac{\pi}{\beta}(2n+1)\,,\\ \omega_{2,n}^* &= \mu_2 + \mathrm{i}\frac{\pi}{\beta}(2n+1)\,,\end{aligned} \tag{3.31}$$

where $n \in \{0, \pm1, \pm2, \pm3, \ldots$ we can solve the integral by using the residue theorem, see also Fig. 3.3 for the integration contour. Finally, we obtain for the integral

$$\begin{aligned}I &= 2\pi\mathrm{i}\lim_{\delta\to\infty}\left\{\mathrm{Res}\, f_1(\omega)\left[1-f_2(\omega)\right]\frac{\delta^2}{\delta^2+\omega^2}\bigg|_{\omega=+\mathrm{i}\delta} + \sum_{n=0}^{\infty}\mathrm{Res}\, f_1(\omega)\left[1-f_2(\omega)\right]\frac{\delta^2}{\delta^2+\omega^2}\bigg|_{\omega=\mu_1+\mathrm{i}\frac{\pi}{\beta}(2n+1)}\right.\\ &\quad\left.+\sum_{n=0}^{\infty}\mathrm{Res}\, f_1(\omega)\left[1-f_2(\omega)\right]\frac{\delta^2}{\delta^2+\omega^2}\bigg|_{\omega=\mu_2+\mathrm{i}\frac{\pi}{\beta}(2n+1)}\right\}\\ &= \frac{\mu_1-\mu_2}{1-e^{-\beta(\mu_1-\mu_2)}}\,,\end{aligned} \tag{3.32}$$

which automatically obeys the simple zero-temperature ($\beta \to \infty$) limit.

With the replacements $\mu_1 \to \mu_R + \Omega$ and $\mu_2 \to \mu_L$ or $\mu_1 \to \mu_L$ and $\mu_2 \to \mu_R - \Omega$, respectively, we can evaluate both terms in the reservoir correlation function, which becomes

$$\gamma^\xi(\Omega) = e^{-\mathrm{i}\xi}\frac{T}{2\pi}\frac{\Omega-V}{1-e^{-\beta(\Omega-V)}} + e^{+\mathrm{i}\xi}\frac{T}{2\pi}\frac{\Omega+V}{1-e^{-\beta(\Omega+V)}}\,, \tag{3.33}$$

where $V = \mu_L - \mu_R$ is the QPC bias voltage. We see that for $\mu_L \gg \mu_R$, only the term with $e^{+\mathrm{i}\xi}$ remains, and $\gamma^\xi(\Omega) \approx e^{+\mathrm{i}\xi}T(\Omega+V)$. In contrast, for $\mu_L \ll \mu_R$, we have $\gamma^\xi(\Omega) \approx e^{-\mathrm{i}\xi}T(\Omega-V)$. This just expresses the fact that for large QPC bias voltage, the current across the QPC becomes unidirectional.

When we now consider the case $\{\Gamma_L, \Gamma_R\} \ll \{TV, |1-\alpha|^2 TV\}$, we approach a bistable system, where for a nearly stationary SET the QPC transmits many charges. Then, the QPC current measured at finite times will be large when the SET dot is empty and reduced otherwise. In this case, the counting statistics approaches the case of telegraph noise. When the dot is empty or filled throughout respectively, the current can easily be determined as

$$I_0 = \frac{T}{2\pi} V \left[\frac{1}{1-e^{-\beta V}} - \frac{1}{e^{+\beta V}-1} \right] = \frac{T}{2\pi} V \,, \qquad I_1 = \left|1-\alpha\right|^2 I_0 \,. \tag{3.34}$$

For finite time intervals Δt, the number of electrons tunneling through the QPC Δn is determined by the probability distribution

$$P_{\Delta n}(\Delta t) = \frac{1}{2\pi} \int\limits_{-\pi}^{+\pi} \mathrm{Tr}\left\{ e^{\mathcal{L}(\xi)\Delta t - \mathrm{i}\Delta n \xi} \rho(t) \right\} d\xi \,, \tag{3.35}$$

where $\rho(t)$ represents the initial density matrix. This quantity can e.g. be evaluated numerically. When Δt is not too large (such that the stationary state is not really reached) and not too small (such that there are sufficiently many particles tunneling through the QPC to meaningfully define a current), a continuous measurement of the QPC current maps to a fixed-point iteration as follows: Now we care about the change of the density matrix when we actually measure a particular change of the particle number. Formally, it would be given by the corresponding propagator

$$\mathcal{P}^{(n)}(\Delta t) = \frac{1}{2\pi} \int\limits_{-\pi}^{+\pi} e^{\mathcal{L}(\xi)\Delta t - \mathrm{i}\Delta n \xi} d\xi \,, \tag{3.36}$$

which has to be applied to the initial density matrix upon measuring a transfer of n particles into the reservoir. Afterwards, we need to renormalize, such that, upon measurement of n particle transfers

$$\rho \to \frac{\mathcal{P}^{(n)}(\Delta t)\rho}{\mathrm{Tr}\left\{\mathcal{P}^{(n)}(\Delta t)\rho\right\}} \,. \tag{3.37}$$

For the generation of a realistic trajectory under continuous monitoring, it is now essential to use the density matrix after the measurement as the initial state for the next iteration. This ensures that e.g. after measuring a large current it is in the next step more likely to measure a large current again than a low current and vice versa. Consequently, the ratio of measured particles divided by measurement time gives a current estimate $I(t) \approx \frac{\Delta n}{\Delta t}$ for the time interval. Such current trajectories are used to track the full counting statistics through quantum point contacts, see Fig. 3.4. In this way, the QPC acts as a detector for the counting statistics of the SET circuit. Finally, we note that for an SET, a QPC only acts as a reliable detector for the occupation of the SET dot. However, when the SET transport is unidirectional (large SET bias), we can infer from this also the number of charges transferred throught the SET.

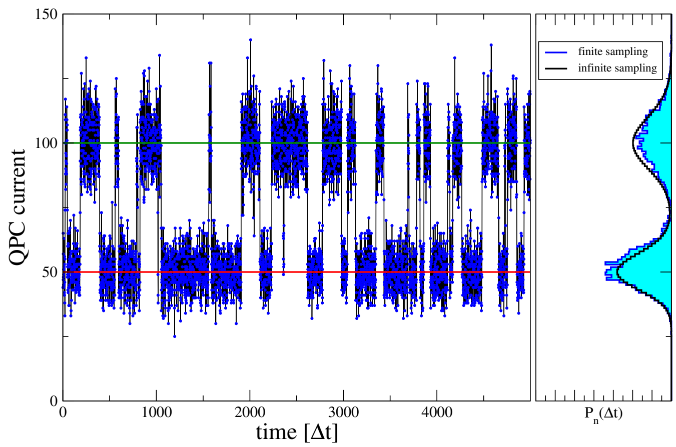

Figure 3.4: Numerical simulation of the time-resolved QPC current for infinite SET and high QPC bias. The QPC current allows to reconstruct the FCS of the SET, since each current blip from low (red line) to high (green line) current corresponds to an electron leaving the SET to its right junction. Parameters: $\Gamma_L \Delta t = \Gamma_R \Delta t = 0.01$, $\beta V \to +\infty$, $TV/(2\pi) = 100.0$, $|1-\alpha|^2 TV/(2\pi) = 50.0$, $f_L = 1.0$, $f_R = 0.0$. The right panel shows the corresponding probability distribution $P_n(\Delta t)$ versus $n = I\Delta t$, where the blue curve is sampled from the finite range of the left panel and the black curve is the theoretical limit for infinitely long times.

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