Standard Model Theory – Tutorial 2

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1. **Dirac algebra**

   Important building blocks when dealing with fermions are the Dirac matrices γ_μ, $\mu = 0, 1, 2, 3$. They fulfil the Clifford algebra

   $$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}\,. \tag{1}$$

   In practical calculations, chains and traces of γ matrices frequently appear. Show by using the defining property (1) and cyclicity of the trace that the following relations hold

   - $\mathrm{Tr}\{\gamma^\mu\} = \mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\} = 0$,
   - $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 4g^{\mu\nu}$,
   - $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\sigma\nu} + g^{\mu\sigma}g^{\rho\nu})$,
   - $\gamma^\mu\gamma_\mu = 4\mathbb{1}$,
   - $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$,
   - $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}\mathbb{1}$.

   In addition we can define

   $$\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{2}$$

   with the property

   $$\{\gamma_5, \gamma_\mu\} = 0,\ \gamma_5^2 = \mathbb{1}\,. \tag{3}$$

   Show that

   - $\mathrm{Tr}\{\gamma_5\} = \mathrm{Tr}\{\gamma_5\gamma^\mu\gamma^\nu\} = 0$,
   - $\mathrm{Tr}\{\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = -4i\epsilon^{\mu\nu\rho\sigma}$.

2. $e^+e^- \to \gamma^* \to \mu^+\mu^-$

   - Write down the matrix element $\mathcal{M}$.
   - The momenta of the incoming and outgoing particles can be parametrized as

     $$p_\pm^\mu = (E, 0, 0, \pm p), \quad q_\pm^\mu = (E, \pm q\sin\theta, 0, \pm q\cos\theta)\,. \tag{4}$$

     Calculate the Mandelstam invariants

     $$s = (p_+ + p_-)^2,\ t = (p_+ - q_+)^2,\ u = (p_+ - q_-)^2\,. \tag{5}$$

- Calculate the differential cross section given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{q}{p}\frac{1}{4}\sum_{\sigma_+,\sigma_-}\sum_{\lambda_+,\lambda_-}|\mathcal{M}|^2$$

  where we have taken the average of the spins of the initial state fermions and the sum over the spins of the outgoing ones.
  Hint: you can use the following results for the spin sums

$$\sum_\lambda u(p,\lambda)\bar{u}(p,\lambda) = \gamma^\mu p_\mu + m\,, \quad \sum_\lambda v(p,\lambda)\bar{v}(p,\lambda) = \gamma^\mu p_\mu - m\,. \tag{6}$$

- Consider the limit $s \gg m_e^2, m_\mu^2$ and calculate the total cross section.