Standard Model Theory – Tutorial 4

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1. **Summary BRS transformations:** Let $\phi(x)$ be a field multiplet which transforms according to a gauge group representation defined by generators T^a. We define $A^\mu = A^\mu_a T^a$, $c = c_a T^a$, and similar for $\bar{c}$, B. The BRS transformations are defined as:

$$\begin{aligned}
sA^\mu(x) &= \partial^\mu c(x) - ig[c(x), A^\mu(x)] \ , & (1)\\
sA^\mu_a(x) &= \partial^\mu c_a(x) + g f_{abc} c_b(x) A^\mu_c(x) \ , & (2)\\
s\phi(x) &= -igc(x)\phi(x) \ , & (3)\\
s\phi_i(x) &= -igc_a(x)T^a_{ij}\phi_j(x) \ , & (4)\\
sc_a(x) &= \frac{1}{2} g f_{abc} c_b(x) c_c(x) \ , & (5)\\
sc(x) &= -igc(x)^2 \ , & (6)\\
s\bar{c}_a(x) &= B_a(x) \ , & (7)\\
sB_a(x) &= 0 \ . & (8)
\end{aligned}$$

s acts as a fermionic differential operator. Show that s is nilpotent, $s^2 X = 0$ for all fields X.

2. **Unitarity of physical S-matrix:**

Consider the process $\psi\bar{\psi} \to A^\mu_a(k_1)A^\nu_b(k_2)$ in a Yang-Mills theory with fermions, e.g. in QCD with one generic quark in the fundamental representation. The indicated colour indices and momenta correspond to the final state vector bosons. The corresponding T-matrix element has the structure

$$T^{(\lambda_1\lambda_2)}_{fi} = \mathcal{M}_{\mu\nu}\epsilon^{\mu *}_1(k_1, \lambda_1)\epsilon^{\nu *}_2(k_2, \lambda_2) \qquad (9)$$

where ϵ_i is the polarization vector of the gauge field i in the final state; these polarization vectors depend on the respective momentum and the indicated polarization states λ_i of the respective vector boson.

Consider a further process $\psi\bar{\psi} \to c_a(k_1)\bar{c}_b(k_2)$ with the same initial state but ghost/antighost in the final state, and denote the T-matrix element as $\mathcal{M}$.

a) Show that the following so-called Slavnov-Taylor identity holds at tree-level:

$$\begin{aligned}
k^\nu_2 \ \mathcal{M}_{\mu\nu} &= \mathcal{M} \ k_{1\mu} & (10)\\
k^\mu_1 \ \mathcal{M}_{\mu\nu} &= \mathcal{M} \ k_{2\nu} & (11)
\end{aligned}$$

(Note there are three Feynman diagrams for $\mathcal{M}_{\mu\nu}$ and one diagram for $\mathcal{M}$. It might be sufficient to do a partial calculation of these Feynman diagrams.)

b) In the computation of probabilities, the sums over polarizations of squares of such amplitudes appear. You can assume the following polarization sums:

$$\text{phys. pol.:} \quad \sum_{\lambda=1,2} \epsilon^{\mu *}(k,\lambda)\epsilon^\rho(k,\lambda) = P^{\mu\rho}(k) \equiv -\left[g^{\mu\rho} - \frac{k^\mu\eta^\rho + \eta^\rho k^\mu}{k\cdot\eta} + \frac{k^\mu k^\rho}{(k\cdot\eta)^2}\right] \qquad (12)$$

$$\text{all pol.:} \quad \sum_{\lambda=1,2,3,4} \epsilon^{\mu *}(k,\lambda)\epsilon^\rho(k,\lambda) = -g^{\mu\rho} \qquad (13)$$

Here η^μ is a constant vector, which can be chosen e.g. as $\eta = (1,0,0,0)$. The result should not depend on this choice. Show:

$$P^{\mu\rho}(k_1)\; P^{\nu\sigma}(k_2)\; \mathcal{M}_{\mu\nu}\; \mathcal{M}^*_{\rho\sigma} = g^{\mu\rho}(k_1)\; g^{\nu\sigma}(k_2)\; \mathcal{M}_{\mu\nu}\; \mathcal{M}^*_{\rho\sigma} - 2\mathcal{M}\; \mathcal{M}^* \tag{14}$$

I.e. in this sense, the unphysical polarizations of the vector fields cancel against the unphysical ghosts. This cancellation is required such that the S-matrix acting only on physical states is unitary. See e.g. Cheng/Li chapter 9.3, Aitchison/Hey chapter 13.5, Peskin/Schroeder chapter 16.3.

3. **Axial gauge:**

   Consider axial gauge, $G^3_a(x) = 0$, i.e. the z-direction is treated in a special way and manifest Lorentz invariance is lost. Show that this gauge is a *physical gauge, in which no ghost fields are required.* You can use either the Faddeev-Popov method or the BRS formalism. Using the latter, you can use the gauge fixing and ghost Lagrangian $\mathcal{L}_{\text{fix,gh}} = s\bar{c}_a G^3_a$. Then you can prove that this leads to the equation of motion $G^3_a = 0$, which can be inserted into the Lagrangian, such that all ghost interaction terms drop out.

   For an account using the Faddeev-Popov method, see Cheng/Li "Gauge theories of elementary particle physics", chapter 9.1.

4. **Degrees of freedom:**

   In classical electrodynamics we know that there are two linearly independent plane wave solutions to Maxwell's equations of the form $A^\mu(x) = \epsilon^\mu(k)e^{-ikx}$ with transverse polarization vectors $\epsilon^\mu(k)$.

   How can we determine the physical degrees of freedom of a theory in the BRS formalism? In this problem, consider the full BRS Lagrangian, however in the limit $g = 0$ (i.e. the free non-interacting theory).

   a) Consider plane wave solutions of the equations of motion $\propto e^{-ikx}$. Show: for a fixed gauge index a and fixed momentum k^μ, there are 6 (not 7 !) linearly independent solutions for the 7 fields A^μ_a, B_a, c_a, $\bar{c}_a$.

   b) Show:

   - two solutions are BRS invariant (i.e. the corresponding fields are annihilated by s in the limit $g = 0$) and *cannot* be written as BRS transforms of something else. These two solutions are by definition the physical degrees of freedom.
   - there is a "BRS quartet" of two BRS non-invariant degrees of freedom and the two corresponding BRS transformations. All of these are by definition unphysical.

Note that this corresponds to the Kugo/Ojima-definition of $Ker(Q)/Im(Q)$ as the physical Hilbert space. For a further exposition of the full BRS+Kugo/Ojima formalism including higher orders see the original paper by Kugo/Ojima or the books by Kugo or Weinberg.