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Relativistic Quantum Field Theory

Exercise 1

Problem 1: (*Euler-Lagrange-Equations*)

a) For the following Lagrange functions, derive the Euler-Lagrange equations and determine the canonical momenta p_I.

i) $L = \dot{q}_1^2 + \dot{q}_2^2$

ii) $L = \dot{q}_1 \dot{q}_2$

iii) $L = (\dot{q}_1 - \dot{q}_2)^2$

Are the second derivatives $\ddot{q}_I$ uniquely determined by the Euler-Lagrange equations? Are the canonical momenta p_I independent?
Calculate in each case the determinant $\det\left(\frac{\partial^2 L}{\partial \dot{q}_I \partial \dot{q}_J}\right)$.

b) If possible, calculate the Hamilton function H for the examples in a) and derive the Hamilton equations of motion. Are the fundamental Poisson brackets $\{q_I, q_J\}_P = \{p_I, p_J\}_P = 0$ and $\{q_I, p_J\}_P = \delta_{IJ}$ well-defined?

Problem 2: (*Vacuum expectation values*)

a) Find the expectation value $\langle 0|\hat{p}^4|0\rangle$, where $\hat{p}$ is the one-dimensional momentum operator. Use only the bosonic commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$ and the effect of the annihilator on the vacuum, $\hat{a}|0\rangle = 0$.

b) Determine the expection value $\langle 0|\hat{\vec{L}}^2|0\rangle$, where $\hat{\vec{L}}$ is the angular momentum operator in three dimensions. Use only the bosonic commutation relations $[\hat{a}_I, \hat{a}_J^\dagger] = \delta_{IJ}$ and $[\hat{a}_I, \hat{a}_J] = [\hat{a}_I^\dagger, \hat{a}_J^\dagger] = 0$ as well as the effect of the annihilator on the vacuum, $\hat{a}_I|0\rangle = 0$.

Problem 3: (*Particle in a magnetic field*)

The Lagrangian function of a particle in an electromagnetic field is given by

$$L = \frac{m}{2}\dot{\vec{r}}^2 + q\dot{\vec{r}} \cdot \vec{A}(t, \vec{r})$$

a) What is the expression for the canonical momentum $\vec{p}$? Derive the equation of motion and the Hamilton function $H(t, \vec{r}, \vec{p})$.

b) Solve the equations of motion for the particular choice $\vec{A} = Bx\vec{e}_y$.

c) What is the Hamilton operator in the special case of b) and what are the conserved quantities? Diagonalize the Hamilton operator and give the spectrum and the associated eigenfunctions. What is the commutator of the velocity operators $\hat{v}_x = d\hat{x}/dt$ and $\hat{v}_y = d\hat{y}/dt$?