Faculty of physics
Prof. Dr. R. Schützhold
Dr. F. Queisser

Relativistic Quantum Field Theory

Exercise 2

Problem 1: (*Field equations*)

The Lagrangian of the electromagnetic field coupled to a current is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu j^\mu = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) - \rho\,\Phi + \vec{j}\cdot\vec{A}$$
$$= \frac{1}{2}\left(\left[\nabla\Phi + \dot{\vec{A}}\right]^2 + \left[\nabla\times\vec{A}\right]^2\right) - \rho\,\Phi + \vec{j}\cdot\vec{A}$$

What are the canonical momenta? Derive the equation of motion. Is the associated action invariant w.r.t. the gauge transformation of the electromagnetic field?

Problem 2: (*Schrödinger field*)

The Lagrangian density of the Schrödinger field is given by

$$\mathcal{L} = i\psi^*(\vec{r})\frac{\partial\psi(\vec{r})}{\partial t} - \frac{1}{2m}\left(\nabla\psi^*(\vec{r})\right)\cdot\left(\nabla\psi(\vec{r})\right) - \psi^*(\vec{r})\psi(\vec{r})V(\vec{r})$$

a) Obtain the canonical momenta.

b) Derive the equations of motion.

c) What is the Hamilton density and the Hamilton function for the Schrödinger field?

Problem 3: (*Coupled harmonic oscillators*)

Consider the following Lagrangian for a system of coupled Harmonic oscillators located at positions $x_n = an$:

$$L = \sum_{n=-\infty}^{\infty} a\left[\frac{\mu}{2}\dot{q}_n^2 - \frac{\kappa}{2}q_n^2 - \frac{\gamma}{2}\frac{\left(q_{n+1}-q_n\right)^2}{a^2}\right]$$

a) Derive the equations of motion for the q_n and solve them with the ansatz

$$q_n = \alpha_p e^{ian\,p - i\omega_p t} + \alpha_p^* e^{-ian\,p + i\omega_p t}\,,$$

where p is any real number and α_n is any complex number. What follows for the relationship between ω_p and p?

b) What is the result in the continuum limit, i.e. in the limiting case $a \to 0$?