TECHNISCHE UNIVERSITÄT DRESDEN

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Relativistic Quantum Field Theory

Exercise 5

Problem 1: (*Schrödinger field with interaction*)

Consider the Schrödinger field with the Hamiltonian ($\hbar = 1$)

$$\hat{H} = \int d^3r \left[\frac{(\nabla\hat{\Psi}^\dagger(\vec{r}))\cdot(\nabla\hat{\Psi}(\vec{r}))}{2m}\right] + \hat{H}_{\text{WW}}$$

and the interaction Hamiltonian

$$\hat{H}_{\text{WW}} = g\int d^3r d^3r' \hat{\psi}^\dagger(\vec{r})\hat{\psi}^\dagger(\vec{r}\,')W(\vec{r}-\vec{r}\,')\hat{\psi}(\vec{r}\,')\hat{\psi}(\vec{r})$$

with the interaction potential $W(\vec{r}-\vec{r}\,')$. The commutation relations for the field operators are the $[\hat{\psi}(\vec{r}), \hat{\psi}^\dagger(\vec{r}\,')] = \delta^{(3)}(\vec{r}-\vec{r}\,')$.

a) Evaluate the energy corrections to the vacuum state $|0\rangle$, the one-particle state $|\vec{p}\rangle$ and the two-particle state $|\vec{p},\vec{k}\rangle$ using stationary perturbation theory. .

b) Calculate the transition between $|\text{in}\rangle = |\vec{k}_1,\vec{k}_2\rangle$ and $|\text{out}\rangle = |\vec{k}_3,\vec{k}_4\rangle$ using time-dependent perturbation theory. .

Problem 2: (*Diagonalization*)

Consider the following Lagrange density in 3+1 dimensions

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)(\partial^\mu\phi) - \kappa\,\vec{r}^{\,2}\phi^2]$$

Derive the Hamiltonian function and the Hamilton operator. What are the eigenfunctions that emerge within the normal mode expansion? Diagonalize the Hamiltonian and compute the ground state energy.

Problem 3: (*Wick theorem*)

A contraction between two operators A and B (which are here either (*bosonic*) annihilation operators or creation operators) is defined as

$$\overbrace{A_1 A_2} = A_1 A_2 - :A_1 A_2:$$

Here, $:AB:$ denotes the normal ordering of operators which means that all annihilation operators are to the right of the creation operators. A string of operators $A_1A_2A_3...$ can be expressed as sum of the normal string plus the normal ordered string after all possible single contractions have been applied plus the normal ordered string after all double contractions have been applied, and so on:

$$A_1A_2... =: A_1A_2... : + \sum_{\text{single}} :\overbrace{A_1A_2}A_3A_4A_5...: + \sum_{\text{double}} :\overbrace{A_1\overbrace{A_2A_3}A_4}A_5...: + \qquad (1)$$

Verify the relation for the strings $\hat{a}_i^\dagger\hat{a}_j\hat{a}_k^\dagger\hat{a}_l$ and $\hat{a}_i\hat{a}_j^\dagger\hat{a}_k\hat{a}_l^\dagger$ and prove the general theorem (1) using complete induction.