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Relativistic Quantum Field Theory

Exercise 9

Addressing issues related to Quantum Field Theory at finite temperatures leads us to shift our focus from vacuum states to thermal states. In this context, the thermal expectation value of an operator, denoted as $\hat{A}$, is defined as follows:

$$\langle \hat{A} \rangle_\beta = \frac{\mathrm{Tr}\left[\hat{A}\exp(-\beta\hat{H})\right]}{\mathrm{Tr}\left[\exp(-\beta\hat{H})\right]} \,.$$

Here, $\beta = 1/(k_B T)$ represents the inverse temperature of the system, $\hat{H}$ signifies the Hamiltonian of the system, and Tr[...] denotes the trace over the entire Fock space. Typically, the trace is computed in the particle-number representation.

Problem 1: (*KMS relation*)

Demonstrate the validity of the following relation for a double-time expectation value

$$\langle \hat{A}(t - i\beta)\hat{B}(t')\rangle_\beta = \langle \hat{B}(t')\hat{A}(t)\rangle_\beta \,.$$

Problem 2: (*Thermal correlator*)

In 3+1 dimensions, consider a non-interacting massless scalar field theory described by the Lagrange density $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. After canonical quantization, compute the thermal correlator

$$\mathrm{Corr}(\vec{r}, t, \vec{r}^{\,\prime}, t') = \langle \hat{\phi}(\vec{r}, t)\hat{\phi}(\vec{r}^{\,\prime}, t')\rangle_\beta \,.$$

For this you can convince yourself that the following relation holds:

$$\langle \hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{p}}\rangle_\beta = \langle \hat{n}_{\vec{k}}\rangle_\beta \delta^{(3)}(\vec{k} - \vec{p}) = \frac{1}{e^{\beta|\vec{k}|} - 1}\delta^{(3)}(\vec{k} - \vec{p}) \,.$$

Show that the correlator can be expressed as a sum of the vacuum contribution and temperature-dependent terms (Hint: write the bose-distribution as geometric series). Determine which terms remain finite in the coincidence limit where $t \to t'$ and $\vec{r} \to \vec{r}^{\,\prime}$.

Problem 3: (*Energy-momentum tensor*)

The thermal expectation value of the energy-momentum tensor can be determined by utilizing the result for $\mathrm{Corr}(\vec{r}, t, \vec{r}^{\,\prime}, t')$ obtained in the previous problem. This involves taking partial derivatives with respect to spatial and temporal variables and then taking the coincidence limit.

Upon closer examination, it becomes evident that the quantity $\langle \hat{T}_{\mu\nu}\rangle_\beta$ is divergent and requires regularization. To address this, one subtracts the part arising from vacuum fluctuations. Proceed to compute the spatial components of the renormalized energy-momentum tensor as

$$\langle (\nabla\hat{\phi}(\vec{r}, t))^2\rangle_{\beta,\mathrm{ren}} = \lim_{\vec{r}\to\vec{r}^{\,\prime}} \langle \nabla\hat{\phi}(\vec{r}, t)\cdot\nabla\hat{\phi}(\vec{r}^{\,\prime}, t)\rangle_\beta - \lim_{\vec{r}\to\vec{r}^{\,\prime}}\lim_{\beta\to\infty} \langle \nabla\hat{\phi}(\vec{r}, t)\cdot\nabla\hat{\phi}(\vec{r}^{\,\prime}, t)\rangle_\beta$$

(Hint: $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$). This procedure is known as point splitting regularization. If you are interested, you may attempt to determine the regularized time component of the energy-momentum tensor.