TECHNISCHE UNIVERSITÄT DRESDEN

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Relativistic Quantum Field Theory

Exercise 10

Problem 1: (*Particle on a sphere*)

Consider a particle governed by the Hamiltonian function given by

$$H = \sum_{i=1}^{D} \frac{p_i^2}{2}$$

subject to the constraint $\sum_{i=1}^{D} x_i^2 = 1$.

a) Identify all constraints and establish their nature (primary or secondary, first or second class).

b) Obtain the Dirac brackets and proceed with the quantization of the system.

Problem 2: (*Schrödinger field*)

Consider the Lagrange density of the Schrödinger field

$$\mathcal{L} = i\psi^*(\vec{r},t)\frac{\partial \psi(\vec{r},t)}{\partial t} - \frac{1}{2m}\left(\nabla\psi^*(\vec{r},t)\right)\cdot\left(\nabla\psi(\vec{r},t)\right) - \psi^*(\vec{r},t)\psi(\vec{r},t)V(\vec{r})$$

a) Find the Hamilton function.

b) Find all constraints and determine their character.

c) Derive the Dirac brackets for ψ and ψ^*.

Problem 3: (*Weyl spinors*)

In the lecture, it was demonstrated that left-handed and right-handed Weyl spinors undergo the following transformations:

$$\psi_L \to \exp\left[i[\vec{\phi} + i\vec{s}\,]\cdot\frac{\vec{\sigma}}{2}\right]\psi_L \quad \text{and} \quad \psi_R \to \exp\left[i[\vec{\phi} - i\vec{s}\,]\cdot\frac{\vec{\sigma}}{2}\right]\psi_R\,.$$

Here, $\vec{\phi}$ represents the rotation angle, and $\vec{s}$ quantifies the direction and magnitude of the boost, where $\cosh(|\vec{s}\,|) = 1/\sqrt{1-v^2/c^2}$ and $\sinh(|\vec{s}\,|) = v/\sqrt{c^2-v^2}$. Utilizing the relativistic energy-momentum relation and the Pauli matrix properties $\sigma_a\sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c$, demonstrate for $\vec{\phi} = 0$ that

$$\psi_L(\vec{p}\,) = \frac{E+m-\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(E+m)}}\psi_L(0) \quad \text{and} \quad \psi_R(\vec{p}\,) = \frac{E+m+\vec{p}\cdot\vec{\sigma}}{\sqrt{2m(E+m)}}\psi_R(0)\,.$$

When $\vec{p} = 0$, the left-handed and right-handed Weyl spinors cannot be distinguished, allowing us to set $\psi_L(0) = \psi_R(0)$. Determine the matrix-valued equation satisfied by the Dirac spinor $\psi = (\psi_R, \psi_L)^T$.