Faculty of physics
Prof. Dr. R. Schützhold
Dr. F. Queisser

Relativistic Quantum Field Theory

Exercise 12

Problem 1: (*Gordon decomposition*)

For the Dirac spinor $u(p)$, give a proof of the identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{(p'+p)^\mu}{2m} + i\sigma^{\mu\nu}\frac{(p'-p)_\nu}{2m}\right]u(p)$$

with $\sigma^{\mu\nu} = i[\gamma^\mu, \gamma^\nu]/2$.

Problem 2: (e^+e^--*annihilation*)

Consider the process $e^+ + e^- \to \mu^+ + \mu^-$. The muons $\mu^\pm$ are described in the same way as $e^\pm$, only with a different mass $m_e \to m_\mu$ and appropriate spinors $u_s^e(p) \to u_s^\mu(p)$ and $v_s^e(p) \to v_s^\mu(p)$.

a) Specify the Feynman diagram(s) of the lowest non-vanishing order.

b) Derive the amplitude M from the Feynman rules.

c) Calculate $|M|^2$ and average/sum $|M|^2$ over the initial/final spin values.

d) Simplify the result using the Casimir trick

$$\sum_{s_1,s_2} \left(\bar{u}_{s_1}(p_1)\Gamma u_{s_2}(p_2)\right)\left(\bar{u}_{s_1}(p_1)\Gamma u_{s_2}(p_2)\right)^* = \mathrm{Tr}\left\{\Gamma(\not{p}_2 + m_2)\bar{\Gamma}(\not{p}_1 + m_1)\right\}$$

where $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$. An analog relation holds for $v_s(p)$ with $\not{p} - m$. Use the traces of the Dirac matrices $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\} = 4g^{\mu\nu}$ and $\mathrm{Tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\} = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$. Traces over an odd number of γ-matrices vanish.

e) How does the process $e^+ + e^- \to e^+ + e^-$ differ from the process $e^+ + e^- \to \mu^+ + \mu^-$? Do not perform any explicit calculation.