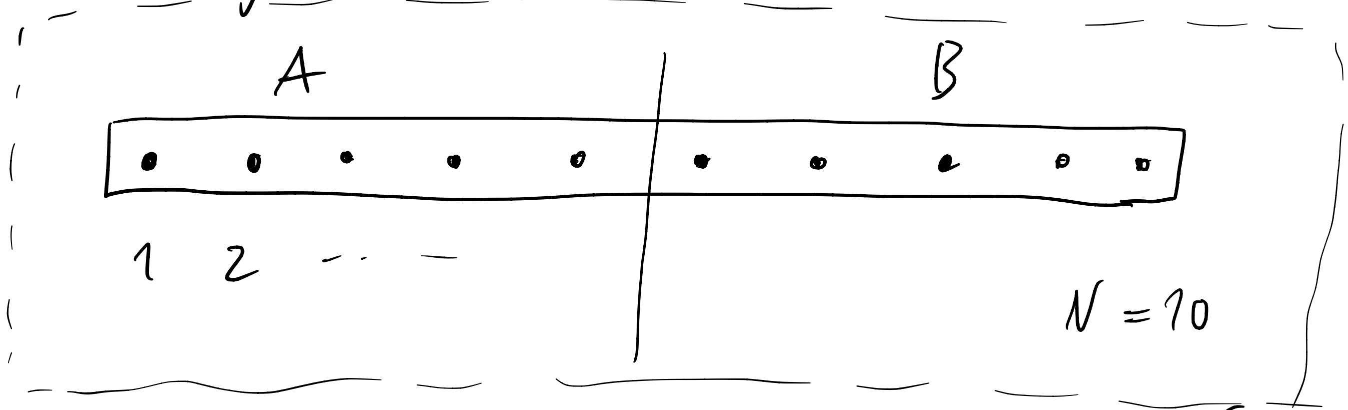
- Elegant solution: Decompose lattice into two subsystems "integer part" $A = \{1, \dots, [\frac{N}{2}]\}$ and $B = \{[\frac{N}{2}]+1, \dots, N\}$



$$I = 2^{[\frac{N+1}{2}]} I_a + I_b \quad \text{with} \quad I_a = \sum_{j=1}^{[\frac{N}{2}]} \sigma_j \, 2^{[\frac{N}{2}]-j}$$

and
$$I_b = \sum_{j=1}^{[\frac{N+1}{2}]} \sigma_{j+[\frac{N}{2}]} \, 2^{[\frac{N+1}{2}]-j} .$$

$\rightsquigarrow$ <u>Crucial advantage</u>: I_a, I_b are small integer $\mathcal{O}(\sqrt{I})$ that we can loop over explicitly.

(# particles in A)

↳ For $n_a = 0, \dots, n$ (assuming $n \leq \frac{N}{2}$), generate all configurations I_a and order from small to big.

↳ Define multiplicity of I_a as $\chi_{I_a} = \binom{[\frac{N+1}{2}]}{n-n_a}$, i.e. as the block dimension of corresponding B-block with $n_b = n - n_a$ $\rightsquigarrow$ Define $J_a(I_a) = \sum_{\tilde{I}_a < I_a} \chi_{\tilde{I}_a}$.

↳ For subsystem B, generate at a given n_b all configurations I_b and order from smaller to bigger.

↳ Define $J_b(I_b)$ for each n_b-block as $J_b(I_b) = \sum_{\tilde{I}_b \leq I_b} 1$.

Then, $\boxed{J = J_a + J_b}$

Example $N = 4$, $n = 2$

	G_a	I_a	J_a	G_b	I_b	J_b	$I = 2^2 I_a + I_b$	$J = J_a + J_b$
$x_{00} = 1$	0 0	0	0	1 1	3	1	3	1
$x_{01} = 2$	0 1	1	1	0 1	1	1	5	2
	0 1	1	1	1 0	2	2	6	3
$x_{10} = 2$	1 0	2	3	0 1	1	1	9	4
	1 0	2	3	1 0	2	2	10	5
$x_{11} = 1$	1 1	3	5	0 0	0	1	12	6

$\leadsto$ Key advantage: Tables $J_{a/b}[I_{a/b}]$ can be explicitly stored for the smaller subsystems A, B, where $I_a, I_b \leq \sqrt{2^N}$,

$\boxed{\text{For } N = 30, \ \sqrt{2^{30}} = 2^{15} = 32\,768}$

Concrete procedure:

(i) For a given I, determine I_a from first $\left\lceil \frac{N}{2} \right\rceil$ bits and I_b from remaining bits,

i.e. $I_a = \left\lfloor I / 2^{\lceil \frac{N+1}{2} \rceil} \right\rfloor$, $I_b = I \bmod 2^{\lceil \frac{N+1}{2} \rceil}$.

(ii) Compute $J = J_a[I_a] + J_b[I_b]$.

- Including translation invariance

Recap: Basics of translation invariance (for simplicity 1D lattice)

- $T_a \hat{x} = \hat{x} + a$. Hamiltonian is translation invariant with period a if $[H, T_a] = 0$.
- Multiple translations form an Abelian group (which is finite for size N and PBC: $T_a^{N+1} = T_a$)

$\leadsto$ irreducible complex representations are one-dimensional and satisfy: $z_{n \cdot a}\, z_{m \cdot a} = z_{(n+m) a}$

$\leadsto$ $z_{na} = (z_a)^n$ $\leadsto$ unitary representation $z_a = e^{ika}$, $k \in [-\frac{\pi}{a}, \frac{\pi}{a}]$ for infinite system. For finite N: $(e^{ika})^N = 1$ implies $k = \frac{2\pi}{N \cdot a} n$, $n = 0, \dots, N-1$.

- Diagonalizing T_a brings H into block-diagonal form, where the blocks correspond to the eigenvalues e^{ika} of T_a. Physical interpretation: k is total momentum.

[from now on: $a=1$, $T_1 = T$]

↳ Since $[T, S_z^{tot}] = 0$, translation symmetry can be implemented for the individual S_z^{tot} blocks (compatible symmetries).

- Find basis states that satisfy $T|\psi\rangle = e^{ik}|\psi\rangle$.

↳ Action of T on computational basis states:

$$T\,|\sigma_1, \sigma_2, \dots, \sigma_{N-1}, \sigma_N\rangle = |\sigma_N, \sigma_1, \sigma_2, \dots, \sigma_{N-1}\rangle.$$

↳ Momentum eigenstates:

$$\boxed{|\sigma_R, k\rangle = \frac{\sqrt{Q_{\sigma_R}}}{N} \sum_{l=0}^{N-1} e^{-ikl}\, T^{l}\, |\sigma_R\rangle}$$

where σ_R is the state with the smallest integer I_{σ_R} within the orbit generated by T, and Q_{σ_R} is the period of that orbit, i.e. $T^{Q_{\sigma_R}} |\sigma_R\rangle = |\sigma_R\rangle$.

Examples $N=4$:

$$\begin{tikzcd} 0011 \arrow[r, "T"] & 1001 \arrow[r, "T"] & 1100 \arrow[r, "T"] & 0110 \arrow[lll, bend left, "T"] \end{tikzcd}$$

i.e. $Q_{0011} = 4$.

Similarly

$$\begin{tikzcd} 0101 \arrow[r, "T"] & 1010 \arrow[l, bend left, "T"] \end{tikzcd}$$

, i.e. $Q_{0101} = 2$

Proof that $|G_R, k\rangle$ is momentum eigenstate

$$T|G_R, k\rangle = \frac{\sqrt{Q_{G_R}}}{N}\sum_{l=0}^{N-1} e^{-ikl}\underbrace{T T^{l}}_{=T^{l+1}}|G_R\rangle =$$

$$= e^{ik}\frac{\sqrt{Q_{G_R}}}{N}\sum_{l=0}^{N-1} e^{-ik(l+1)} T^{l+1}|G_R\rangle =$$

$l+1 \to l$

$$= e^{ik}\frac{\sqrt{Q_{G_R}}}{N}\sum_{l=1}^{N} e^{-ikl} T^{l}|G_R\rangle =$$

$e^{ikN} = e^{ik0}$

$T^{N} = T^{0}$

$$= e^{ik}\underbrace{\frac{\sqrt{Q_{G_R}}}{N}\sum_{l=0}^{N-1} e^{-ikl} T^{l}|G_R\rangle}_{\overset{\text{Def.}}{=}|G_R, k\rangle} = e^{ik}|G_R, k\rangle. \square$$

$\hookrightarrow$ $|6_R, k\rangle$ is only admissible if $k = n\frac{2\pi}{N}$ satisfies

$$\boxed{Q_{6_R} \cdot n \mod N = 0} \quad \text{otherwise } e^{ikQ_{6_R}} \neq 1.$$