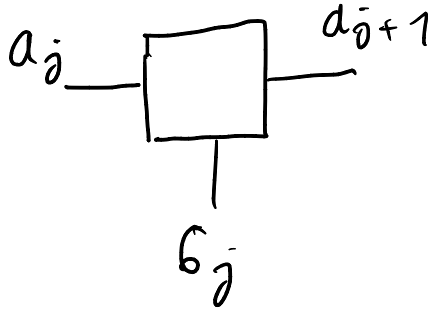
- Tensor network states in 1D: Matrix product states (MPS)



$\simeq A^{\sigma_j}_{a_j, a_{j+1}}$ with $a_j \in 1, \dots, \chi_j$ and $a_{j+1} \in 1, \dots, \chi_{j+1}$

$\rightsquigarrow |\Psi_w\rangle \overset{\text{PBC}}{=} \sum_{\sigma_1, \dots, \sigma_N} A^{\sigma_1}_{a_1, a_2} A^{\sigma_2}_{a_2, a_3} \cdots A^{\sigma_N}_{a_N, a_1} |\sigma_1, \dots, \sigma_N\rangle$ (summing over repeated indices)

[χ's known as bond dimensions]

- Set of matrices $\{A^{\sigma_j}\}$ represents the set of variational parameters (Not all parameters are independent as $A^{\sigma_j} \to S_j A^{\sigma_j} S^{-1}_{j+1}$ $\forall j$ leaves $|\Psi_w\rangle = \sum_{\{\sigma\}} \mathrm{Tr}(A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_N}) |\sigma_1, \dots, \sigma_N\rangle$ unchanged $\rightsquigarrow$ gauge degree of freedom).

- Basic entanglement properties of MPS:

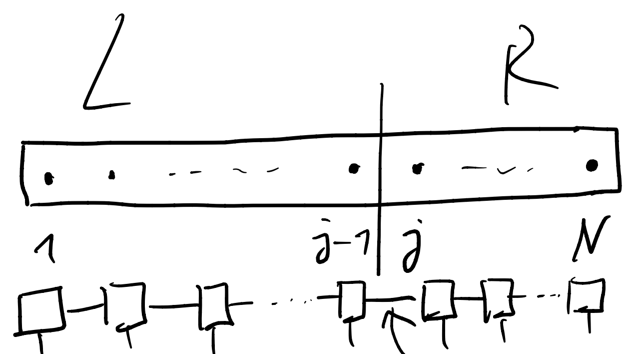
↳ If all A^{σ_j} were numbers ($\chi_j = 1$), we would have:

$$|\psi_w\rangle = \left(\sum_{\sigma_1} A^{\sigma_1}|\sigma_1\rangle\right) \otimes \left(\sum_{\sigma_2} A^{\sigma_2}|\sigma_2\rangle\right) \otimes \cdots \otimes \left(\sum_{\sigma_N} A^{\sigma_N}|\sigma_N\rangle\right)$$

(non-entangled product state).

↳ For $\chi_j \times \chi_{j+1}$ matrices A^{σ_j}, when dividing the system

as



, we have a cut at

the summation index $a_j = 1, \dots, \chi_j$, and the

total state can be written as (OBC!)

$$|\Psi_w\rangle = \sum_{a_j=1}^{\chi_j} \lambda_{a_j} \underbrace{|L_{a_j}\rangle}_{L} \otimes \underbrace{|R_{a_j}\rangle}_{R}$$

[possible to achieve by gauge choice]

$$\langle L_{a_j} | L_{a_k}\rangle = \delta_{j,k}$$

$\rightsquigarrow$ Maximum entanglement entropy:

$$S_L = -\mathrm{Tr}(\rho_L \log \rho_L) \quad \text{with} \quad \rho_L = \mathrm{Tr}_R\left(|\Psi_w\rangle\langle\Psi_w|\right) = \sum_{a_j=1}^{\chi_j} |\lambda_{a_j}|^2 |L_{a_j}\rangle\langle L_{a_j}|$$

[$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$, $|\Psi\rangle \in \mathcal{H}$, $\rho = |\Psi\rangle\langle\Psi|$, $\rho_L = \mathrm{Tr}_R\, \rho$, with

$|\Psi\rangle = \sum_{i,j} \Psi_{ij} \underbrace{|i\rangle}_{L} \otimes \underbrace{|j\rangle}_{R}$. $\rho_L = \Psi\Psi^+$, i.e. $\rho_L^{ij} = \Psi_{ie}\Psi^+_{ej}$]

For a given bond dimension χ_j, we have

$$\boxed{S_L \leq \log \chi_j}$$

"=" for $|\lambda_{a_j}|^2 = \frac{1}{\chi_j}$

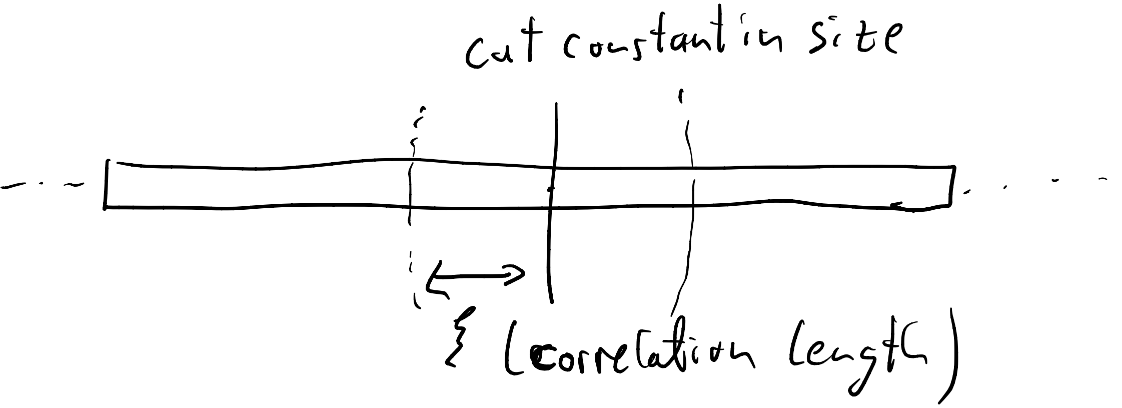
- Problem: Generic state $|\psi\rangle \in \mathcal{H}$ (say random state) exibits extensive entanglement scaling, i.e. $S_C \sim$ length of subsystem (volume law).

$\rightsquigarrow \chi_{N/2} \geq e^{\frac{N}{2} \cdot \text{const}}$, i.e. χ exponential in system size .

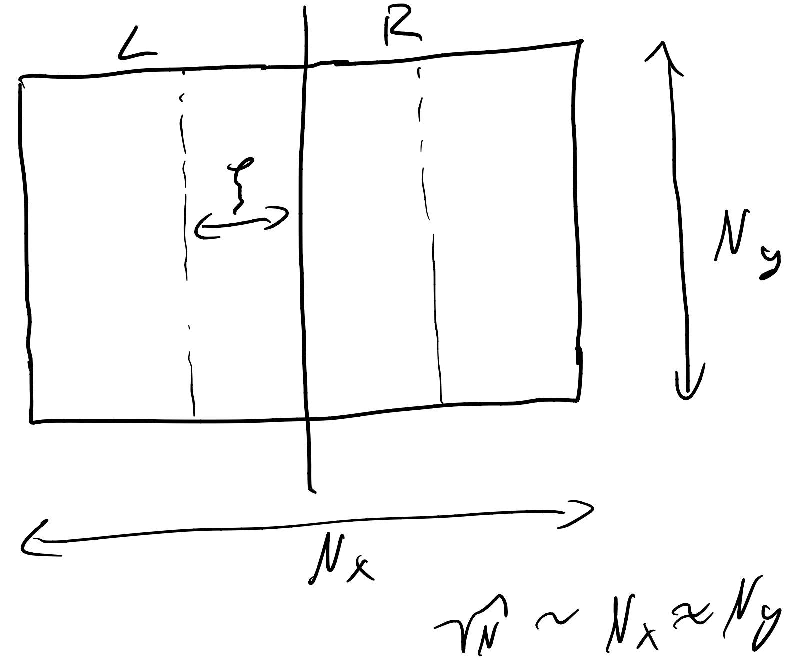
- Physical fact to the rescue of the TNS ansatz:

↳ Ground states of (gapped) local Hamiltonians only exhibit <u>area law</u> entanglement scaling, i.e. $S_C \sim$ area of boundary between subsystems (size of cut)

⇝ S_L independent of N in 1D



⇝ $S_L \sim \sqrt{N}$ in 2D



↳ Critical systems exhibit <u>logarithmic</u> corrections to to the area law entanglement (polynomial increase in x with system size). (see, e.g. Eisert et al. RMP 82, 277).

⇝ Limited entanglement of many physically relevant states justifies the TNS ansatz.

<u>Rule of thumb</u>: TNS best for low-temperature equilibrium physics in low dimensions.