- More simple but common MPS operations:

↳ Overlap $\langle \phi | \psi \rangle$ of MPS $|\psi\rangle = \sum_{\{\sigma\}} M^{\sigma_1} \dots M^{\sigma_N} |\sigma_1, \dots, \sigma_N\rangle$

and MPS $|\phi\rangle = \sum_{\{\sigma\}} \tilde{M}^{\sigma_1} \dots \tilde{M}^{\sigma_N} |\sigma_1, \dots, \sigma_N\rangle$

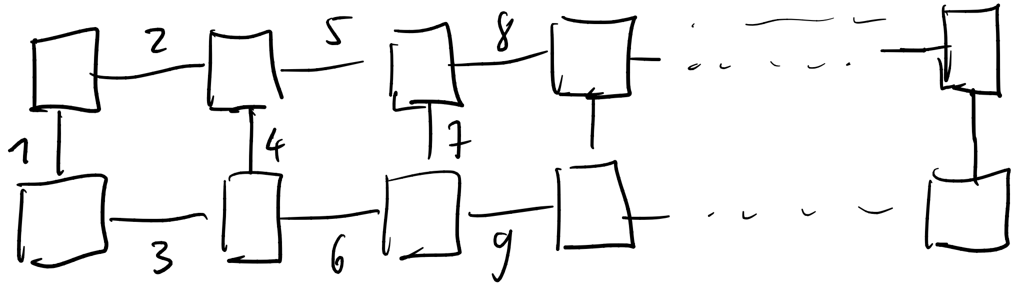
$$\leadsto \langle \phi | \psi \rangle = \sum_{\{\sigma\}} \tilde{M}^{\sigma_1 *} \dots \tilde{M}^{N *} M^{\sigma_1} \dots M^{\sigma_N} = \sum_{\{\sigma\}} \tilde{M}^{\sigma_N \dagger} \dots \tilde{M}^{\sigma_1 \dagger} M^{\sigma_1} \dots M^{\sigma_N} =$$

"transpose" scalar $M^{\sigma_1 *} \dots M^{\sigma_N *}$

$$= \sum_{\sigma_N} \tilde{M}^{\sigma_N \dagger} \dots \underbrace{\left(\sum_{\sigma_2} \tilde{M}^{\sigma_2 \dagger} \underbrace{\left(\sum_{\sigma_1} \tilde{M}^{\sigma_1 \dagger} M^{\sigma_1} \right)}_{\text{perform first}} M^{\sigma_2} \right)}_{\text{perform second}} \dots M^{\sigma_N}.$$

⋮

Graphical rep.:



$\langle\phi|$

$|\psi\rangle$

(labeling indicates order of contraction)

Cost: $(2N-1)\cdot d$ $(\chi\times\chi)$-matrix multiplications.

- Matrix element of simple tensor product operators (MPOs of bond dim 1)

$$O = O^{(1)}\otimes \cdots \otimes O^{(N)} \rightsquigarrow \langle\phi|O|\psi\rangle =$$

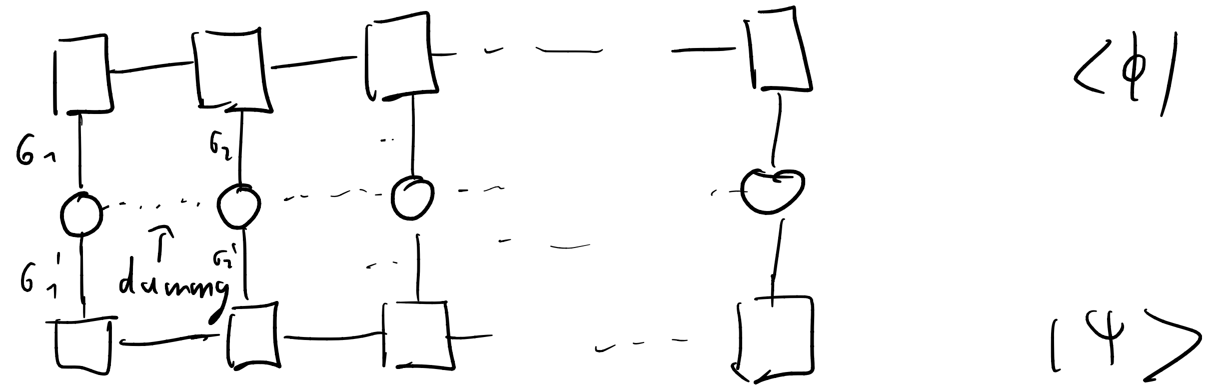
$$= \sum_{\{\sigma,\sigma'\}} \tilde{M}^{\sigma_1 *}\cdots\tilde{M}^{*\sigma_N}\, O^{\sigma_1\sigma_1'}\cdots O^{\sigma_N\sigma_N'}\, M^{\sigma_1'}\cdots M^{\sigma_N'} =$$

numbers

[transpose $\tilde{M}^{\sigma_1 *}\cdots\tilde{M}^{*\sigma_N}$ and regroup numbers $O^{\sigma_i\sigma_i'}$]

$$= \sum_{\sigma_N,\sigma_N'} O^{\sigma_N\sigma_N'}\tilde{M}^{\dagger\sigma_N}\Big(\cdots\Big(\sum_{\sigma_2\sigma_2'} O^{\sigma_2\sigma_2'}\tilde{M}^{\dagger\sigma_2}\Big(\sum_{\sigma_1\sigma_1'} O^{\sigma_1\sigma_1'}\tilde{M}^{\dagger\sigma_1}M^{\sigma_1'}\Big)M^{\sigma_2'}\cdots\Big)M^{\sigma_N'}$$

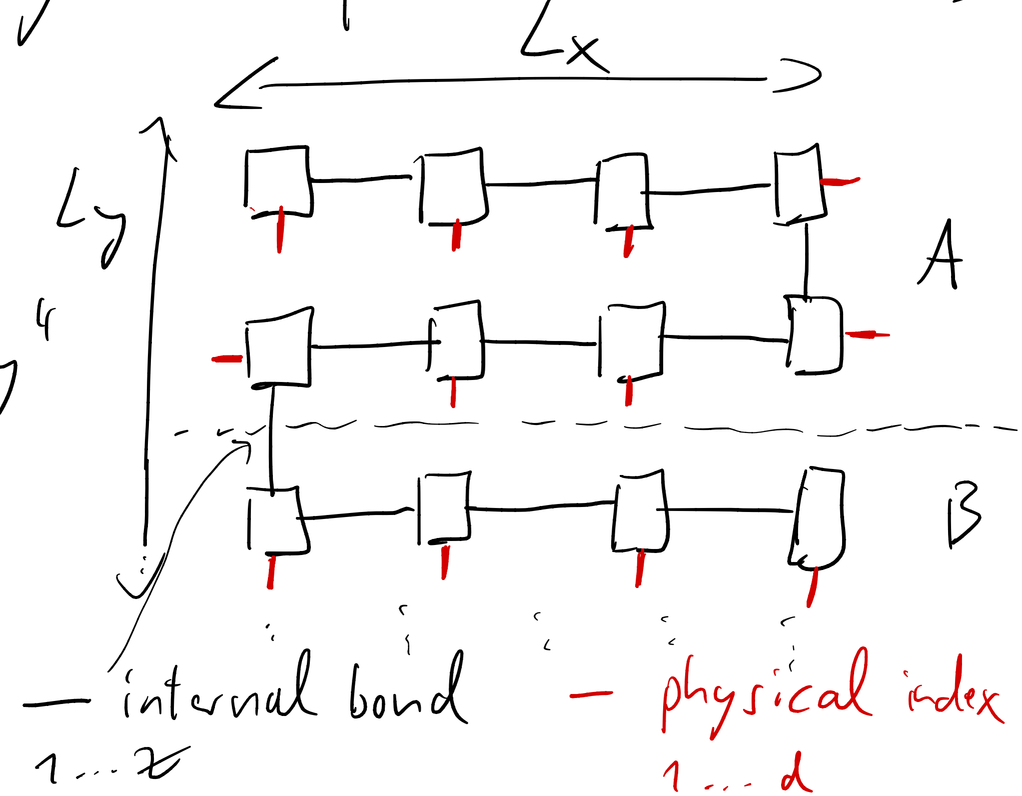
↳ Graphical rep.:



C.4: Going to Higher Spatial Dimensions

- Simplest approach:

2D MPS in "snake geometry"

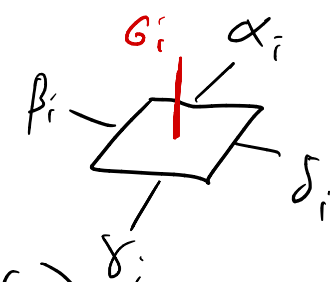


- Pros:
  - ↳ Tensor network connectivity still 1D-like
    ⇝ simple contraction, i.e. efficient exact calculation of state amplitudes and observables.
  - ↳ Successful in practice and easy to implement

from existing DMRG code.

- Cons:
  - ↳ Entanglement entropy $S \sim L_x$ (area law)
    $\rightsquigarrow$ bond dimension $\chi \sim e^{\alpha L_x}$ required.
  - ↳ Mostly suitable for ribbon and cylinder geometries, since aspect ratio $L_y / L_x \gg 1$ preferable.

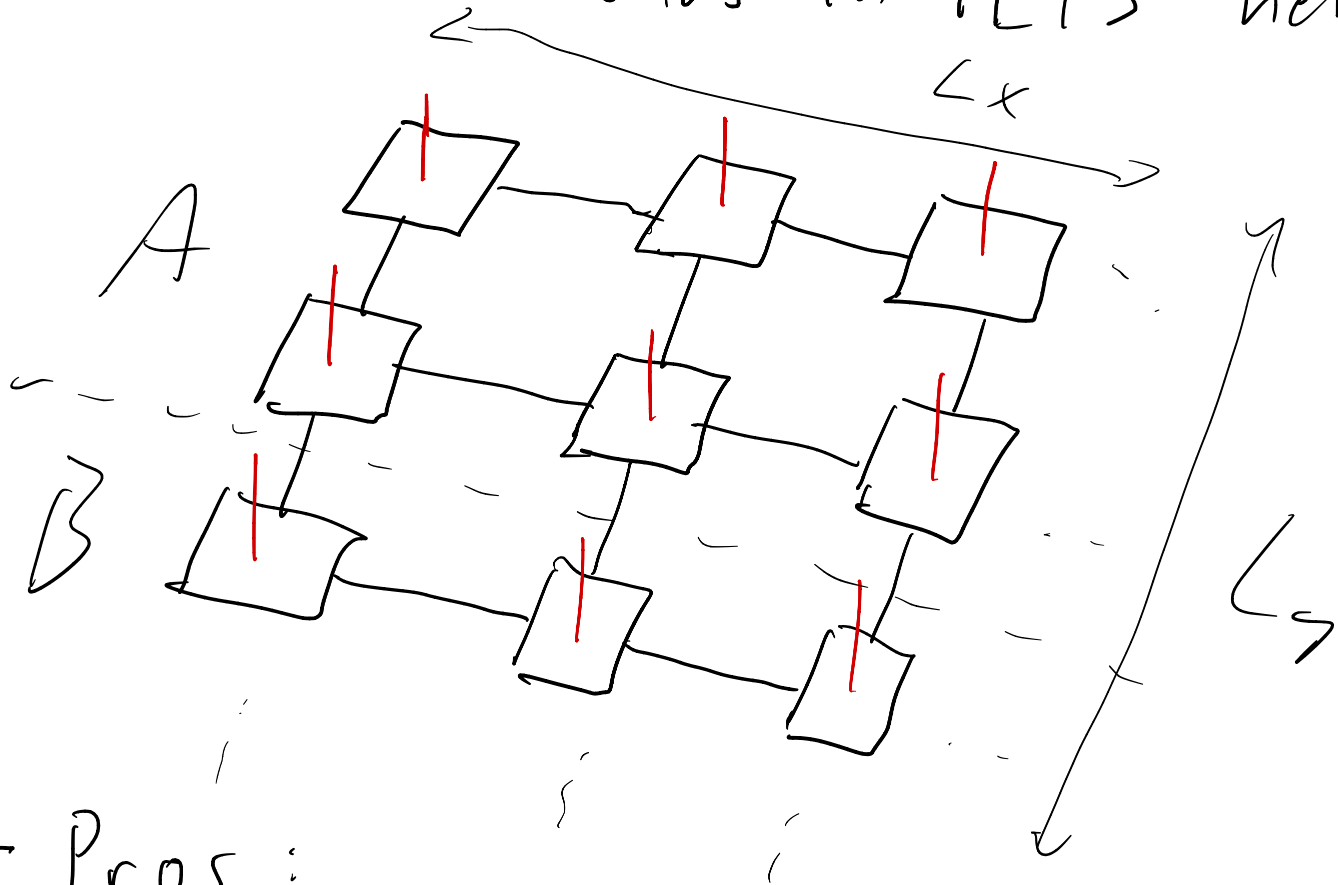
• Genuine 2D approach:
Projected entangled pair states (PEPS)

- General form of PEPS: 

$\simeq M^{\sigma_i}_{\alpha_i \beta_i \gamma_i \delta_i}$

$$|\psi\rangle = \sum_{\{\sigma\}} \underbrace{M^{\sigma_1} \cdots M^{\sigma_N}}_{\text{all lower indices summed over, according to shared}} |\sigma_1, \ldots, \sigma_N\rangle$$

bonds in PEPS network:



- Pros:
  - ↳ Area-law entanglement scaling now respected at finite bond dimension
  - ↳ Exact solution of several QMB problems known in the form of low bond-dim PEPS,

e.g. Kitaev toric code (see PRL 96, 220601)

- Cons: ↳ Exact contraction numerically hard

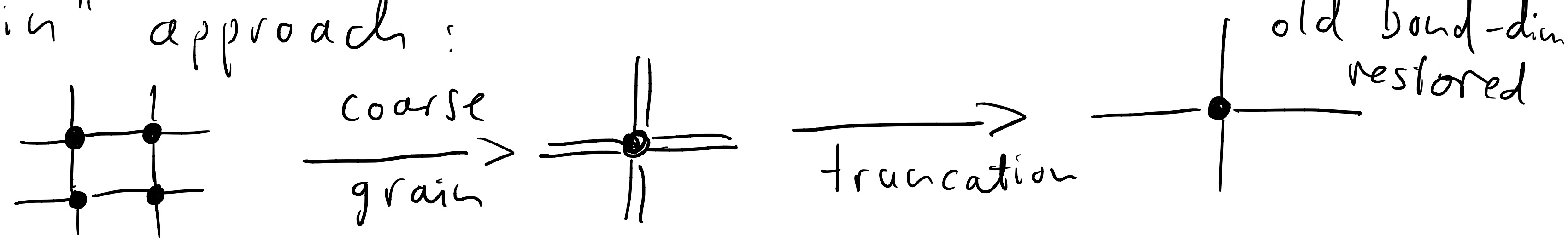
$\left(\sim e^{\alpha\sqrt{N}}\right)$ ← total # sites

⇝ approximate schemes for contraction necessary!

- Approximate contraction of PEPS

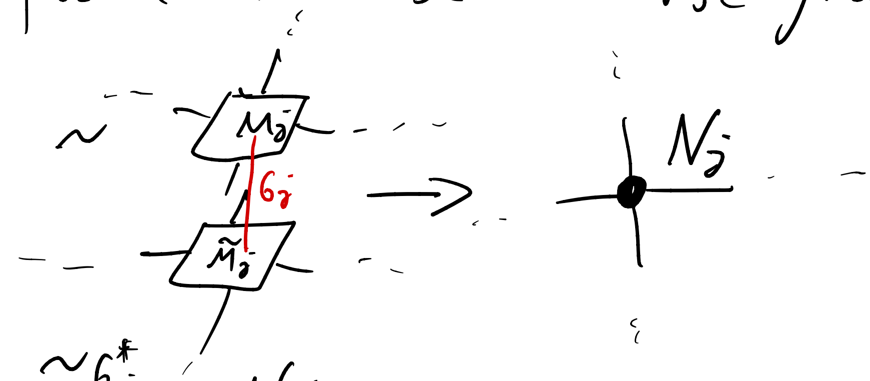
- Tensor entanglement renormalization group approach:

↳ Iteratively coarse grain 2D network using "block-spin" approach:



↳ Physical sites must be contracted first as they are not supposed to be coarse grained:

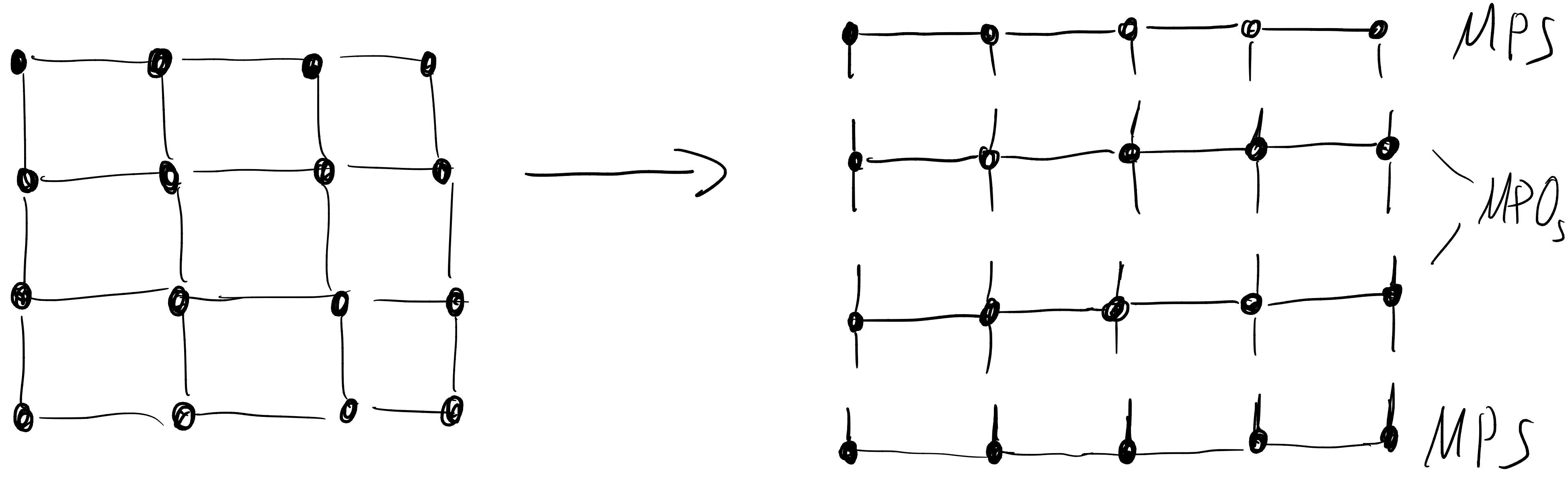
⇝ Example $\langle \phi | \psi \rangle \sim$



with $N_j = \sum_{\sigma_j} \tilde{M}^{\sigma_j^*} \otimes M^{\sigma_j}$

↳ Crucial issue: Truncation of tensors will generically be a lossy compression, unless bond dimension is grown exponentially.

– Contraction by reducing PEPS network (with physical indices contracted) to MPS language:



↳ Successively apply MPOs to MPS "row by row".

↳ Issue: Bond-dim. of MPS grows exponentially with the number of rows (applied MPOs)

⇝ Truncate ... (lossy compression) ..

- Upshot on TNS methods in 2D:
  - ↳ MPS have clear 2D analog with PEPS
  - ↳ Area law entanglement makes exact TNS methods exponentially costly in one direction of the 2D system.
    - ⇝ "unfair advantage" of DMRG in 1D is partly lost, but 2D MPS and PEPS still powerfull and competitive methods for many 2D QMB systems.