Die Navier-Stokes-Gleichungen

Darstellung in kartesischen Koordinaten

$$\rho\cdot\left(\frac{\partial}{\partial t}\vec{v}+\vec{v}\bullet\nabla\vec{v}\right)=\rho\cdot\overrightarrow{f_m}-\nabla p+\eta\cdot\Delta\vec{v}+\left(\zeta+\frac{1}{3}\cdot\eta\right)\cdot\nabla\left(\nabla\bullet\vec{v}\right)$$

wobei: $\zeta=\lambda+\frac{2}{3}\cdot\eta$ (ζ = Druck-/Volumen-/Komressions-/dilationale/ zweite Zähigkeit; λ = erste Lamé-Konstante)

Masse ∗ (lokale + konvektive Beschleunigung) = Raumkräfte + Druckkräfte + Widerstand zu Formänderung + Widerstand zur Volumenänderung

Gleichungen für inkompressible Fluide:

x-Richtung: $$\rho\cdot\left(\frac{\partial v_x}{\partial t}+v_x\cdot\frac{\partial v_x}{\partial x}+v_y\cdot\frac{\partial v_x}{\partial y}+v_z\cdot\frac{\partial v_x}{\partial z}\right)=\rho\cdot f_x-\frac{\partial p}{\partial x}+\eta\cdot\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

y-Richtung: $$\rho\cdot\left(\frac{\partial v_y}{\partial t}+v_x\cdot\frac{\partial v_y}{\partial x}+v_y\cdot\frac{\partial v_y}{\partial y}+v_z\cdot\frac{\partial v_y}{\partial z}\right)=\rho\cdot f_y-\frac{\partial p}{\partial y}+\eta\cdot\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)$$

z-Richtung: $$\rho\cdot\left(\frac{\partial v_z}{\partial t}+v_x\cdot\frac{\partial v_z}{\partial x}+v_y\cdot\frac{\partial v_z}{\partial y}+v_z\cdot\frac{\partial v_z}{\partial z}\right)=\rho\cdot f_z-\frac{\partial p}{\partial z}+\eta\cdot\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

Darstellung in Zylinderkoordinaten (für inkompressible Fluide)

r-Richtung: $$\rho\cdot\left(\frac{\partial v_r}{\partial t}+v_r\cdot\frac{\partial v_r}{\partial r}+\frac{v_\phi}{r}\cdot\frac{\partial v_r}{\partial \phi}+v_z\cdot\frac{\partial v_r}{\partial z}-\frac{{v_\phi}^2}{r}\right)=\rho\cdot f_r-\frac{\partial p}{\partial r}+\eta\cdot\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 v_r}{\partial \phi^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\cdot\frac{\partial v_\phi}{\partial \phi}\right)$$

θ-Richtung: $$\rho\cdot\left(\frac{\partial v_\phi}{\partial t}+v_r\cdot\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi}{r}\cdot\frac{\partial v_\phi}{\partial \phi}+v_z\cdot\frac{\partial v_\phi}{\partial z}+\frac{v_r\cdot v_\phi}{r}\right)=\rho\cdot f_\phi-\frac{1}{r}\cdot\frac{\partial p}{\partial \phi}+\eta\cdot\left(\frac{\partial^2 v_\phi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_\phi}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{\partial^2 v_\phi}{\partial z^2}-\frac{v_\phi}{r^2}+\frac{2}{r^2}\cdot\frac{\partial v_r}{\partial \phi}\right)$$

z-Richtung: $$\rho\cdot\left(\frac{\partial v_z}{\partial t}+v_r\cdot\frac{\partial v_z}{\partial r}+\frac{v_\phi}{r}\cdot\frac{\partial v_z}{\partial \phi}+v_z\cdot\frac{\partial v_z}{\partial z}\right)=\rho\cdot f_z-\frac{\partial p}{\partial z}+\eta\cdot\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 v_z}{\partial \phi^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

Darstellung in Kugelkoordinaten (für inkompressible Fluide)

r-Richtung:

$$\rho\cdot\left(\frac{\partial v_r}{\partial t}+v_r\cdot\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\cdot\sin\theta}\cdot\frac{\partial v_r}{\partial \phi}-\frac{v_q^{2}+v_\phi^{2}}{r}\right)=\rho\cdot f_r-\frac{\partial p}{\partial r}+\eta\cdot\left[\Delta v_r-\frac{2\cdot v_r}{r^2}-\frac{2}{r^2\cdot\sin^2\theta}\cdot\frac{\partial\left(v_\theta\cdot\sin\theta\right)}{\partial\theta}-\frac{2}{r^2\cdot\sin\theta}\cdot\frac{\partial v_\phi}{\partial\phi}\right]$$

Θ-Richtung:

$$\rho\cdot\left(\frac{\partial v_\theta}{\partial t}+v_r\cdot\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\cdot\sin\theta}\cdot\frac{\partial v_\theta}{\partial \phi}+\frac{v_r\cdot v_\theta}{r}-\frac{v_\phi^{2}\cdot\cot\theta}{r}\right)=\rho\cdot f_\theta-\frac{1}{r}\cdot\frac{\partial p}{\partial \theta}+\eta\cdot\left(\Delta v_\theta+\frac{2}{r^2}\cdot\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2\cdot\sin^2\theta}-\frac{2\cdot\cos\theta}{r^2\cdot\sin^2\theta}\cdot\frac{\partial v_\phi}{\partial\phi}\right)$$

φ-Richtung:

$$\rho\cdot\left(\frac{\partial v_\phi}{\partial t}+v_r\cdot\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\cdot\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\cdot\sin\theta}\cdot\frac{\partial v_\phi}{\partial \phi}+\frac{v_r\cdot v_\phi}{r}+\frac{v_\theta\cdot v_\phi\cdot\cot\theta}{r}\right)=\rho\cdot f_\phi-\frac{1}{r}\cdot\frac{\partial p}{\partial \phi}+\eta\cdot\left(\Delta v_\varphi+\frac{2}{r^2\cdot\sin\theta}\cdot\frac{\partial v_r}{\partial\phi}+\frac{2\cdot\cos\theta}{r^2\cdot\sin^2\theta}\cdot\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi}{r^2\cdot\sin^2\theta}\right)$$

Die Differentialoperatoren

kartesische Koordinaten:

Nabla - Operator:
(nur für kartes. Koordinaten)

$$\nabla=\vec{e}_x\cdot\frac{\partial}{\partial x}+\vec{e}_y\cdot\frac{\partial}{\partial y}+\vec{e}_z\cdot\frac{\partial}{\partial z}$$

Laplace - Operator:

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

Gradient eines Skalars:

$$\text{grad}(s)=\nabla s=\vec{e}_x\cdot\frac{\partial}{\partial x}s+\vec{e}_y\cdot\frac{\partial}{\partial y}s+\vec{e}_z\cdot\frac{\partial}{\partial z}s$$

Divergenz eines Vektors:

$$\text{div}\left(\vec{v}\right)=\nabla\bullet\vec{v}=\frac{\partial}{\partial x}v_x+\frac{\partial}{\partial y}v_y+\frac{\partial}{\partial z}v_z$$

Divergenz eines Gradienten:

$$\text{div}(\text{grad}(s))=\Delta s=\nabla\bullet\nabla s=\frac{\partial^2}{\partial x^2}s+\frac{\partial^2}{\partial y^2}s+\frac{\partial^2}{\partial z^2}s$$

Kugelkoordinaten:

$$\text{grad}(s)=\vec{e}_r\cdot\frac{\partial}{\partial r}s+\vec{e}_\theta\cdot\frac{1}{r}\cdot\frac{\partial}{\partial\theta}\cdot s+\vec{e}_\phi\cdot\frac{1}{r\cdot\sin\theta}\cdot\frac{\partial}{\partial\phi}s$$

$$\text{div}\left(\vec{v}\right)=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot v_r\right)+\frac{1}{r\cdot\sin\theta}\cdot\frac{\partial}{\partial\theta}\cdot\left(\sin\theta\cdot v_\theta\right)+\frac{1}{r\cdot\sin\theta}\cdot\frac{\partial v_\phi}{\partial\phi}$$

$$\Delta=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\cdot\frac{\partial}{\partial r}\right)+\frac{1}{r^2\cdot\sin\theta}\cdot\frac{\partial}{\partial\theta}\cdot\left(\sin\theta\cdot\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\cdot\sin^2\theta}\cdot\frac{\partial^2}{\partial\phi^2}$$

Literatur:

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