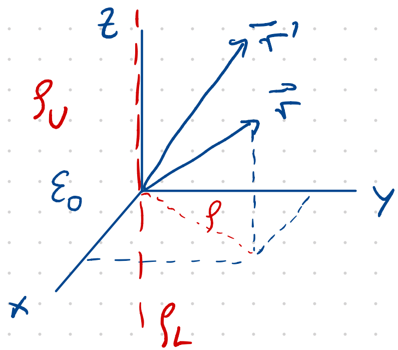
Elektrisches Skalarpotential einer unendlich langen homogenen Linienladung



ohne Randbedingungen

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_V \frac{\rho_V(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \, d^3 r'$$

$$\rho_V(\vec{r}\,') = \rho_L \, \delta(x'-0)\,\delta(y'-0) = \rho_L \, \delta(x')\,\delta(y')$$

$$|\vec{r} - \vec{r}\,'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \frac{\rho_L \, \delta(x')\,\delta(y')}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \, dx'\,dy'\,dz'$$

$$= \frac{1}{4\pi\varepsilon_0} \int\limits_{-\infty}^{\infty} \rho_L \frac{1}{\sqrt{\underbrace{x^2+y^2}_{\rho^2} + (z-z')^2}} \, dz' = \frac{\rho_L}{4\pi\varepsilon_0} \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{\rho^2 + (z-z')^2}} \, dz'$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \int\limits_{z-\infty}^{z+\infty} \frac{1}{\sqrt{\rho^2 + u^2}} \, du$$

Substitution:
$u = z - z'$
$du = -dz'$
o. Grenze: $z - \infty$
u. Grenze: $z + \infty$

$$\phi(\vec{r}) = \frac{\rho_L}{4\pi\varepsilon_0} \int_{z-\infty}^{z+\infty} \frac{1}{\sqrt{\rho^2+u^2}}\, du$$

Für jedes beliebige, aber endliche z gilt: $\phi(\vec{r}) = \frac{\rho_L}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\rho^2+u^2}}\, du$

$\Rightarrow$ $\phi(\vec{r})$ ist unabhängig von z

Stammfunktion:

$$\int \frac{1}{\sqrt{a^2+x^2}}\, dx \overset{192}{=} \ln\left(x+\sqrt{a^2+x^2}\right) + C$$

$$\phi(\vec{r}) = \frac{\rho_L}{4\pi\varepsilon_0} \Big[\ln\big(u+\sqrt{\rho^2+u^2}\big)\Big]_{-\infty}^{\infty} = \frac{\rho_L}{4\pi\varepsilon_0} \ln\left[\frac{\infty+\sqrt{\rho^2+\infty^2}}{-\infty+\sqrt{\rho^2+\infty^2}}\right] ??$$

alternativ:

$$\phi(\vec{r}) = \frac{2\cdot\rho_L}{4\pi\varepsilon_0} \Big[\ln\big(u+\sqrt{\rho^2+u^2}\big)\Big]_{0}^{\infty} = \frac{\rho_L}{2\pi\varepsilon_0} \ln\left[\frac{\infty+\sqrt{\rho^2+\infty^2}}{0+\sqrt{\rho^2+0^2}}\right] = \infty ??$$

für alle ρ ???

Könnte es daran liegen, dass Ladung unendlich lang ist?

$\longrightarrow$ endlich lange Linienladung $-L \leq z \leq L$

$$\phi(\vec{r}) = \frac{\rho_L}{4\pi\varepsilon_0} \int_{z-L}^{z+L} \frac{1}{\sqrt{\rho^2+u^2}}\, du$$

für endliche Linienladung hängt ϕ von z ab!

<u>Betrachten um $z=0$:</u>

$$\phi(\vec{r})\Big|_{z=0} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{1}{\sqrt{\rho^2+u^2}}\, du = \frac{\rho_L}{4\pi\varepsilon_0} \ln\left[\frac{L+\sqrt{\rho^2+L^2}}{-L+\sqrt{\rho^2+L^2}}\right] = \frac{\rho_L}{2\pi\varepsilon_0}\left[\frac{L+\sqrt{\rho^2+L^2}}{\rho}\right]$$

$$= \frac{\rho_L}{4\pi\varepsilon_0} \ln\left[\frac{\sqrt{\frac{\rho^2}{L^2}+1}+1}{\sqrt{\frac{\rho^2}{L^2}+1}-1}\right] = \frac{\rho_L}{2\pi\varepsilon_0} \ln\left[\frac{\sqrt{\left(\frac{\rho}{L}\right)^2+1}+1}{\left(\frac{\rho}{L}\right)}\right]$$

$$\phi(\vec r)\Big|_{z=0} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{\sqrt{\left(\frac{\rho}{L}\right)^2+1}+1}{\left(\frac{\rho}{L}\right)}\right] = \frac{\rho_L}{2\pi\varepsilon_0}\left\{\ln\left(\sqrt{\left(\frac{\rho}{L}\right)^2+1}+1\right)-\ln\left(\frac{\rho}{L}\right)\right\}$$

Weiterhin: – Limes $L\to\infty$ $\rightsquigarrow$ $\phi(\vec r)\Big|_{z=0} = \infty$ für alle $\rho<\infty$

– $L<\infty$: $\rho\to\infty$ ist problematisch $\phi = \infty - \infty$??

<u>Ausweg</u>: Skalarpotential ist nur bis auf additive Konstante bestimmt.

$$\underbrace{\phi(\vec r)\Big|_{z=0} - \phi(\vec r)\Big|_{\substack{z=0\\ \rho=\rho_0}}}_{\tilde\phi(\vec r)} = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{\left(1+\sqrt{\left(\frac{\rho}{L}\right)^2+1}\right)}{\left(\frac{\rho}{L}\right)}\cdot\frac{\left(\frac{\rho_0}{L}\right)}{\left(1+\sqrt{\left(\frac{\rho_0}{L}\right)^2+1}\right)}\right]$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{\frac{\rho_0}{\rho}\left(1+\sqrt{\left(\frac{\rho}{L}\right)^2+1}\right)}{1+\sqrt{\left(\frac{\rho_0}{L}\right)^2+1}}\right]$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{\frac{\rho_0}{\rho}+\sqrt{\left(\frac{\rho_0}{L}\right)^2+\left(\frac{\rho_0}{\rho}\right)^2}}{1+\sqrt{\left(\frac{\rho_0}{L}\right)^2+1}}\right]$$ ist endlich für $\rho\to\infty$

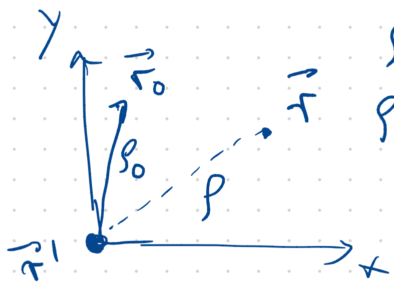
Nun: $\lim\limits_{L\to\infty} \tilde{\Phi}(z) = \lim\limits_{L\to\infty}\left[\frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\frac{\rho_0}{\rho}\left(1+\sqrt{\left(\frac{\rho}{L}\right)^2+1}\right)}{1+\sqrt{\left(\frac{\rho_0}{L}\right)^2+1}}\right)\right] = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho_0}{\rho}\right)$

$$= -\frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho}{\rho_0}\right)$$

Greensche Funktion?

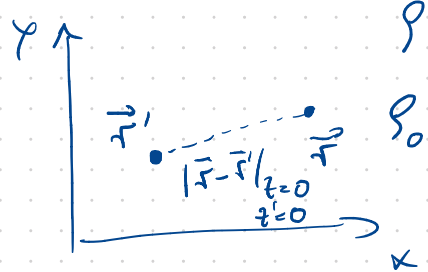
$$\Delta\tilde{\phi} = \Delta\left[-\frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho}{\rho_0}\right)\right] = -\frac{1}{\varepsilon_0}\rho_V(\vec{r}) = -\frac{1}{\varepsilon_0}\rho_L\,\delta(x-0)\,\delta(y-0)$$

$$\Delta G(\vec{r},\vec{r}\,') = -\frac{1}{\varepsilon_0}\delta(\vec{r}-\vec{r}\,') = -\frac{1}{\varepsilon_0}\delta(x-x')\,\delta(y-y')\,\cancel{\delta(z-z')}$$



$\rho = \sqrt{x^2+y^2}$

$\rho_0 = \sqrt{x_0^2+y_0^2}$



$\rho \to \sqrt{(x-x')^2+(y-y')^2}$

$\rho_0 \to \sqrt{(x_0-x')^2+(y_0-y')^2}$

$$G(\vec{r},\vec{r}\,') = -\frac{1}{2\pi\varepsilon_0}\ln\frac{\sqrt{(x-x')^2+(y-y')^2}}{\sqrt{(x_0-x')^2+(y_0-y')^2}} = -\frac{1}{2\pi\varepsilon_0}\ln\frac{|\vec{r}-\vec{r}\,'|_{z=0,z'=0}}{|\vec{r}_0-\vec{r}\,'|_{z=0,z'=0}}$$

für Linienladung entlang der z-Achse:

$$G(\vec{r},\vec{r}\,') = -\frac{1}{2\pi\varepsilon_0}\ln\left(\frac{\rho}{\rho_0}\right)$$