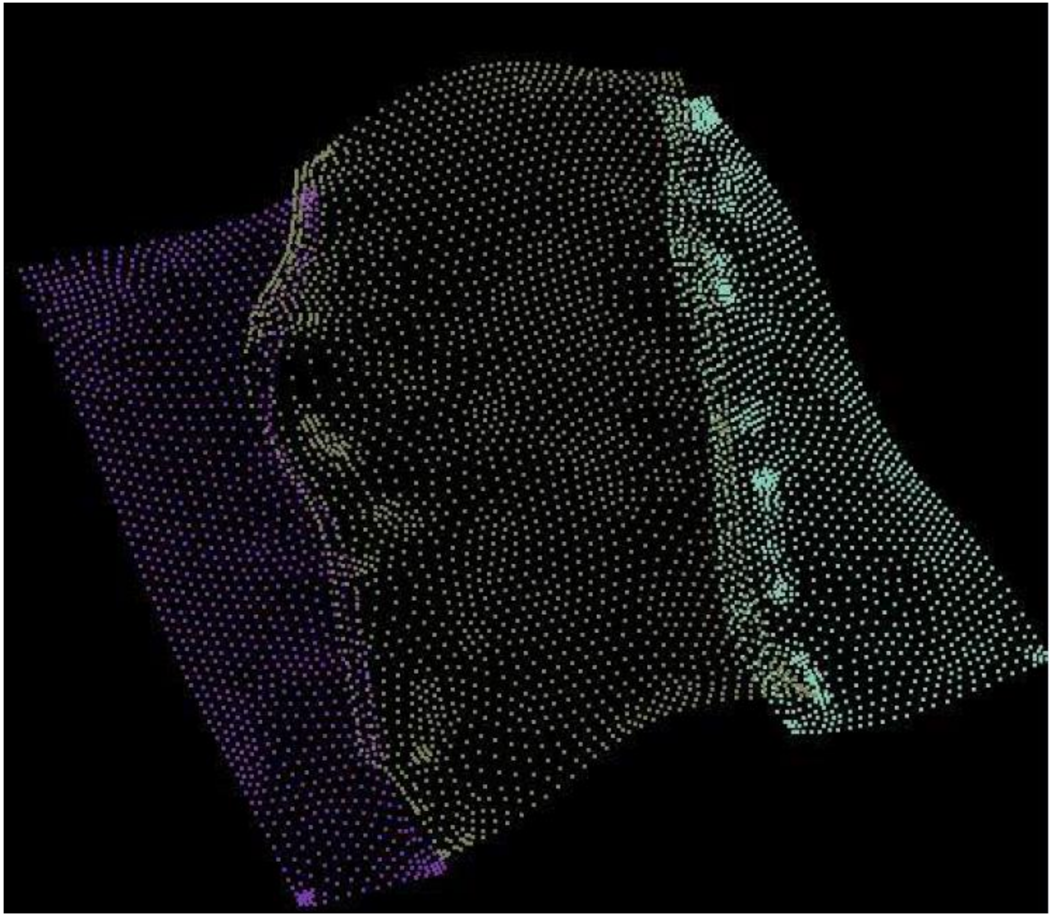
Übersicht

1. Definitionen, Funktionen, Anwendungen (Vorlesung 1)
2. Koordinatensysteme und -transformationen (Vorlesung 2+3)
3. Räumliche Datenmodellierung (Vorlesung 4 - 6)
4. **Vermaschungen**
5. Räumliche Interpolation
6. Transformationen, Filtermethoden, Sonstiges

Zerlegungen und Triangulierung
(Partition and Triangulation)



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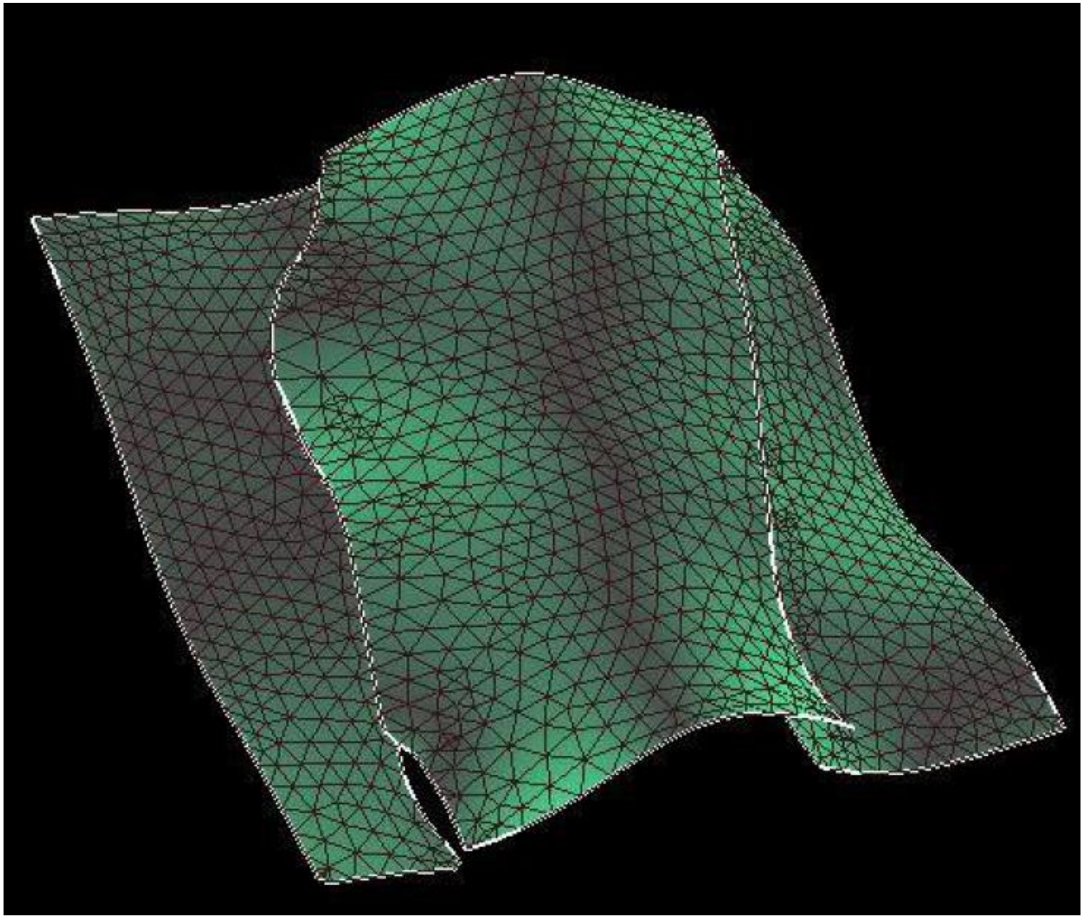
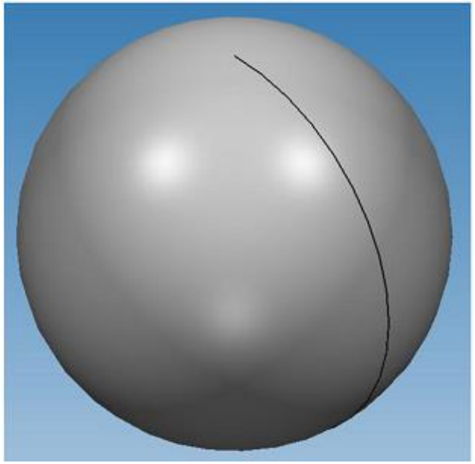
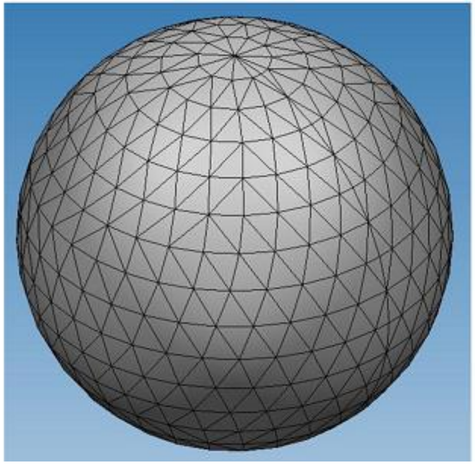


Zerlegungen und Triangulierung (Partition and Triangulation)

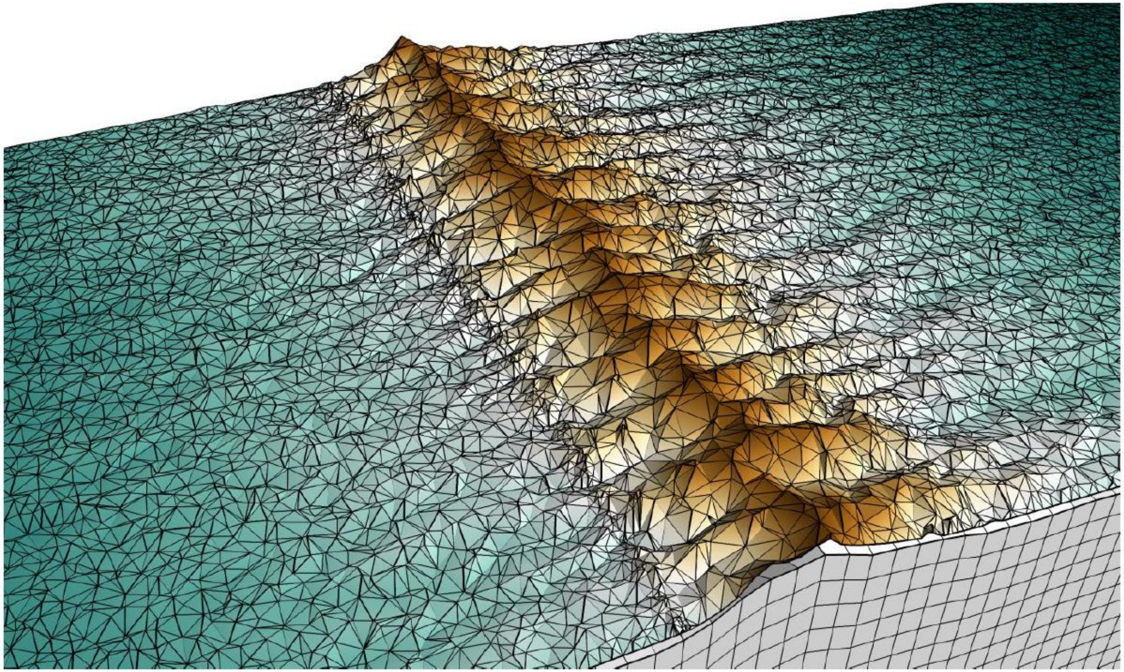
Im Bereich “computer aided geometric design” oder “computational geometry” bezieht sich der Terminus “Zelegung” auf jedes beliebige Verfahren, um ein beliebiges n-dimensionales Objekt in eine Menge benachbarter polytopaler* n-dimensionaler Zellen zu zerlegen.





Zerlegungen und Triangulierung
(Partition and Triangulation)

Im Kontext von GIS werden Zerlegungen verwendet, wenn ein Geoobjekt über seine Bestandteile beschrieben werden soll. Diesen Bestandteilen können dadurch eigene Attribute zugewiesen werden.



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Auf diesen Bestandteilen lassen sich Berechnungen zumeist sehr viel einfacher durchführen, als auf dem Objekten selbst. Eine digitale visuelle Darstellung eines beliebigen Objektes ist ohne eine Vermaschung des Objektes in einfache diskrete Zellen oft nicht möglich.

Zerlegungen sind eine spezielle Form des Vektormodells.

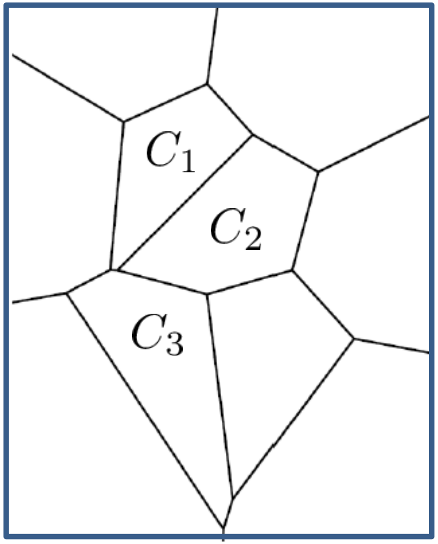
Partition / Tesselation

A **Partition** of a region $\Omega \subset \mathbb{R}^n$ is a set of open cells $C_i \subset \mathbb{R}^n$, $i = 1, \ldots, N$, with $C_i \cap C_j = \emptyset$, $i \neq j$, such that

$$\Omega = \bigcup_{i=1,\ldots,N} C_i$$

Partitions are also called **Tesselations**.

- Partition → Zerlegung
- Tesselation → Vermaschung



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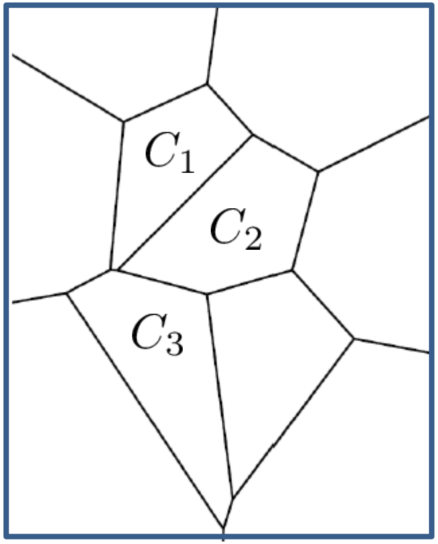
Partitions are also called **Tesselations** (Vermaschungen).

The **openeness** needs to be considered for a precise mathematical formulation $\longrightarrow$ later!

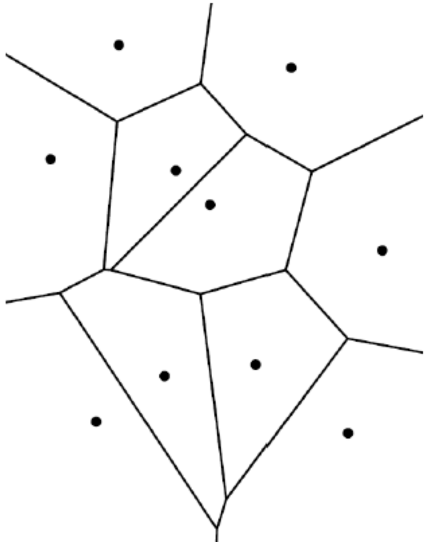


Voronoi Tesselation

Let $\{p_i \in \mathbb{R}^n : i = 1, \dots, N\}$ be a set of points. A **Voronoi cell** V_i denotes the set

$$V_i = \{x \in \mathbb{R}^n : |x - p_i| < |x - p_j|, \, j \neq i\}.$$

Such a cell is a particular form of **Polyhedron** (or polygon, in 2D). Furthermore, the cells $C_i = V_i$, $i = 1, \dots, N$ "nearly" form a partition of $\mathbb{R}^n$.



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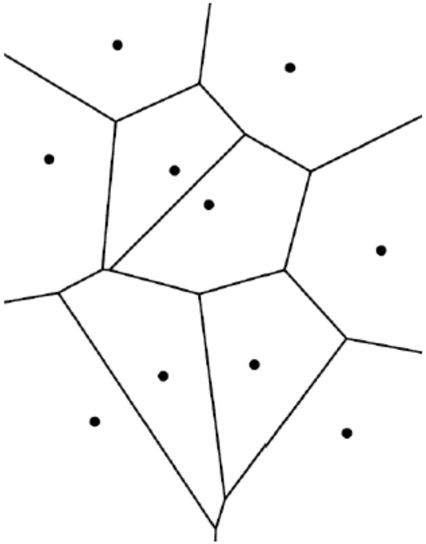
Why just "nearly"? Formally, also the edges of the Voronoi cells need to be included, yielding further cells C_i, $i = N + 1, \ldots, N + M$, as well as the points at the intersections of edges, yielding yet further cells C_i, $i = N + M + 1, \ldots, N + M + K$. This is a result of the openness condition.



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The **bisector** of two points p_i, p_j is denoted by

$$B_{i,j} = \{x \in \mathbb{R}^n : |x - p_i| = |x - p_j|\}$$

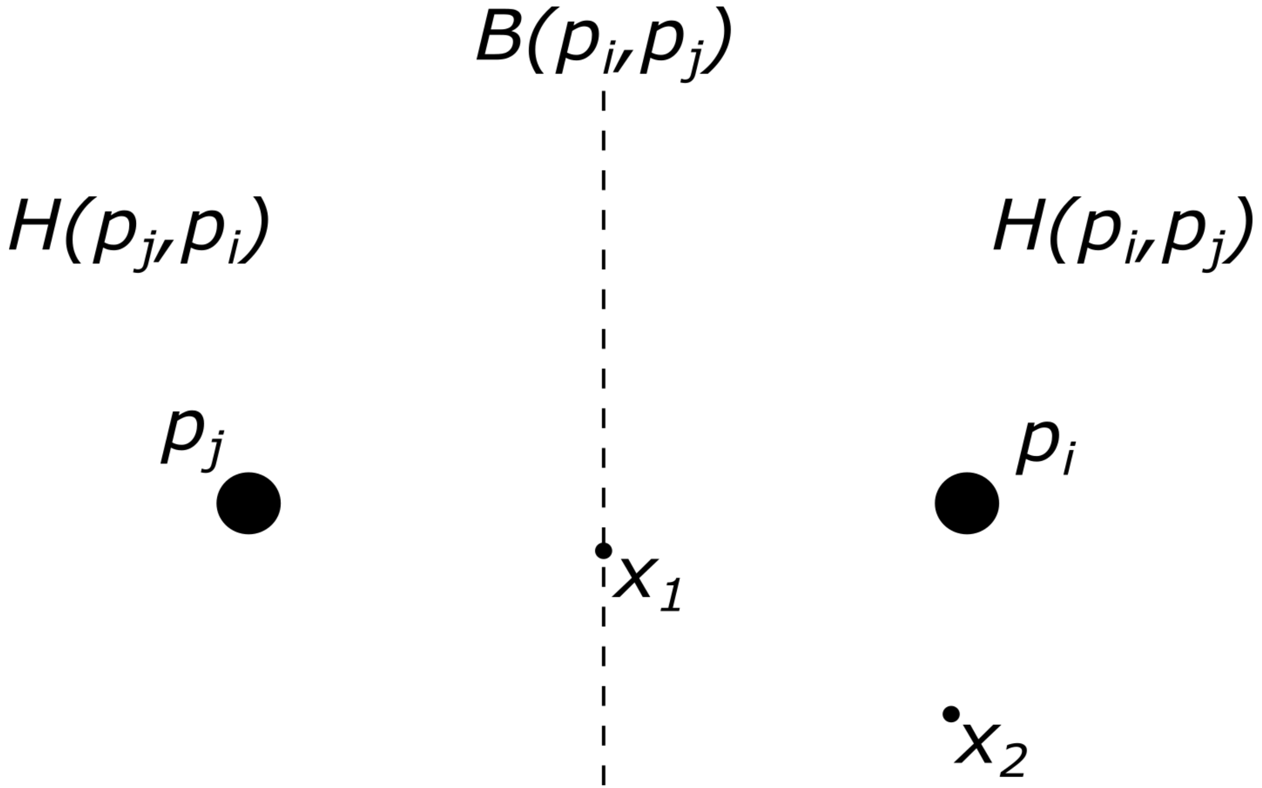
and separates $\mathbb{R}^n$ into two half-spacee. In particular, it represents the boundary of the half-space

$$H_{i,j} = \{x \in \mathbb{R}^n : |x - p_i| < |x - p_j|\}.$$

Each Voronoi cell can then be expressed in the form

$$V_i = \bigcap_{\substack{j=1,\ldots,N, \\ i \neq j}} H_{i,j}.$$

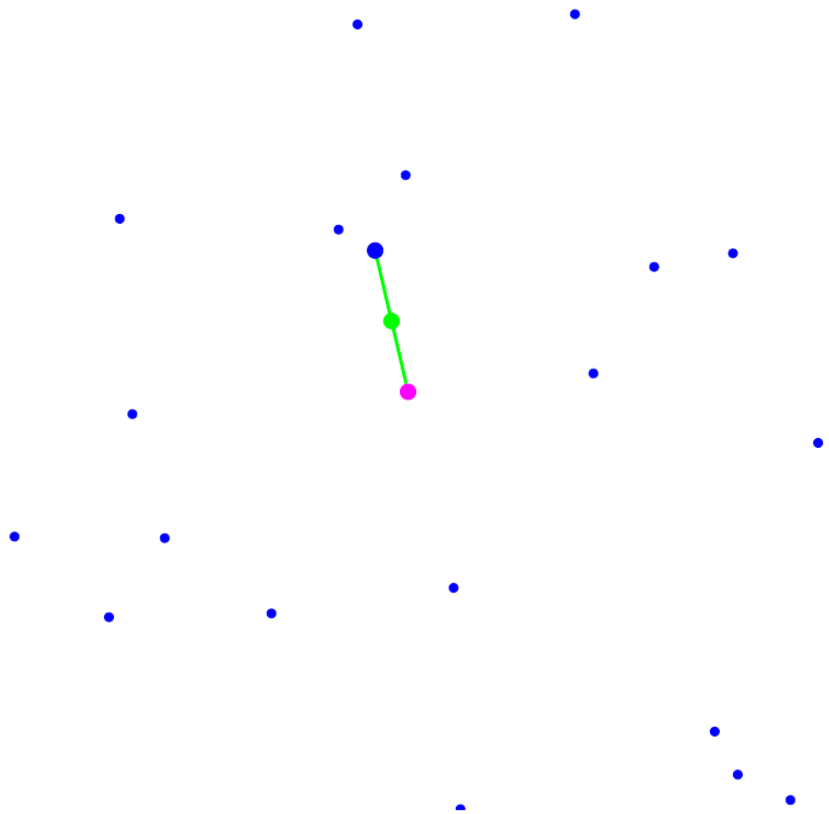
Voronoi Tesselation



$$|x_1 - p_i| = |x_1 - p_j| \rightarrow x_1 \in B(p_i, p_j)$$

$$|x_2 - p_i| < |x_2 - p_j| \rightarrow x_2 \in H(p_i, p_j)$$

Example: Voronoi Tesselation in 2-D



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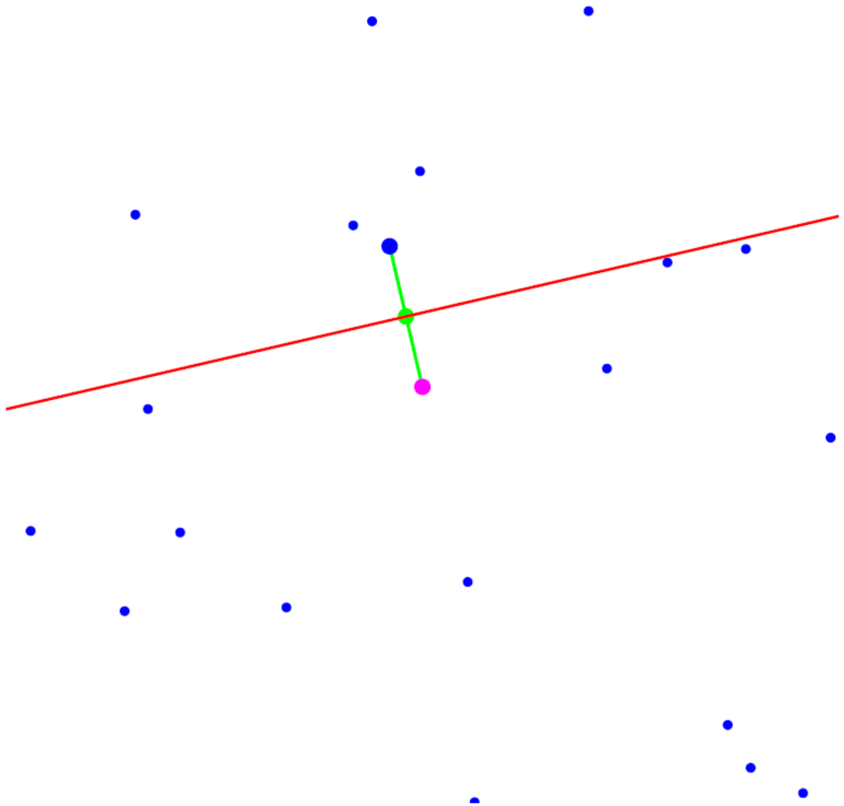

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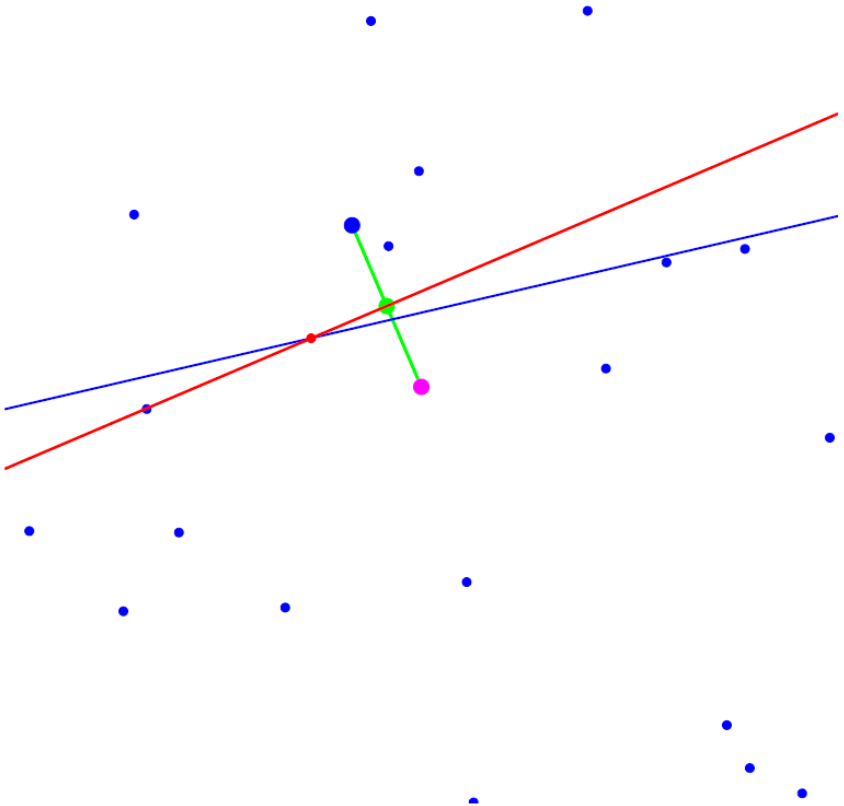

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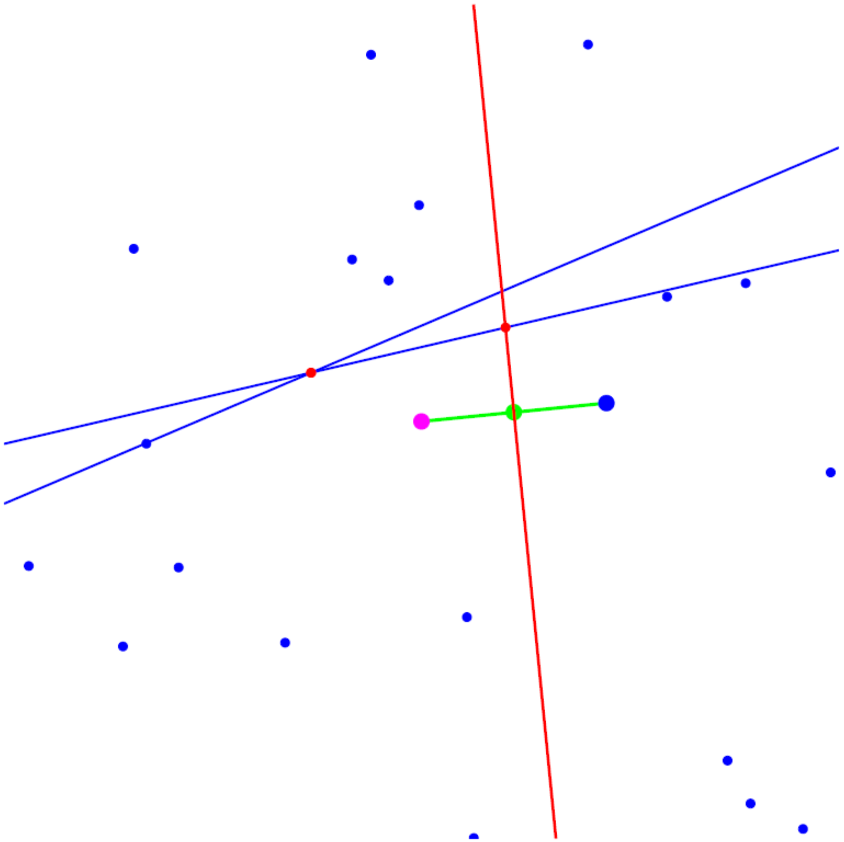

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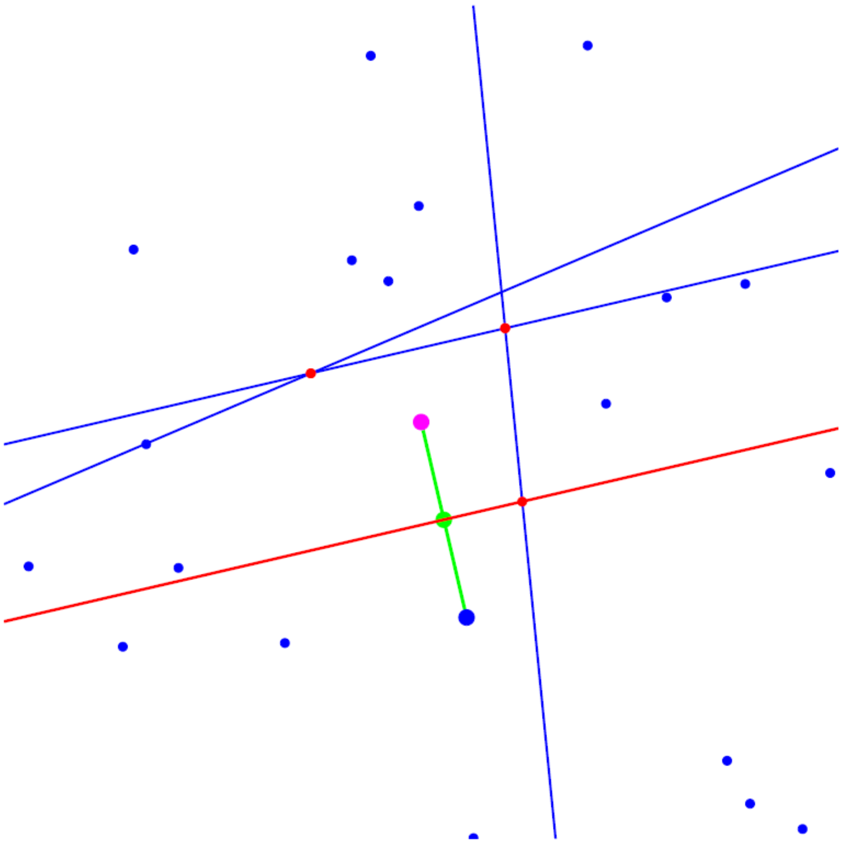

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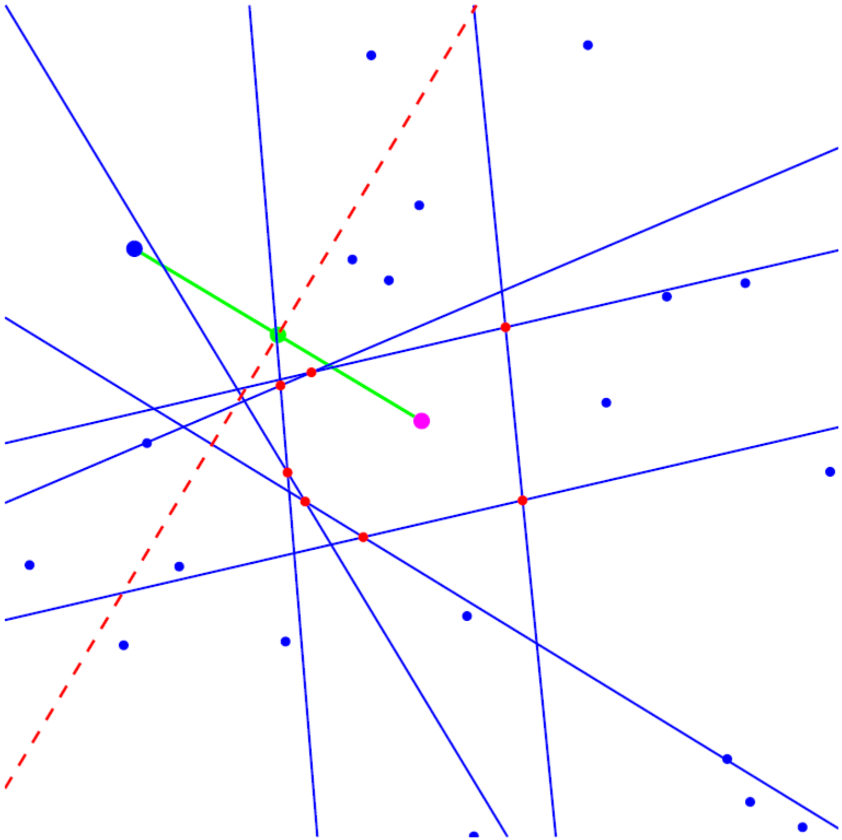

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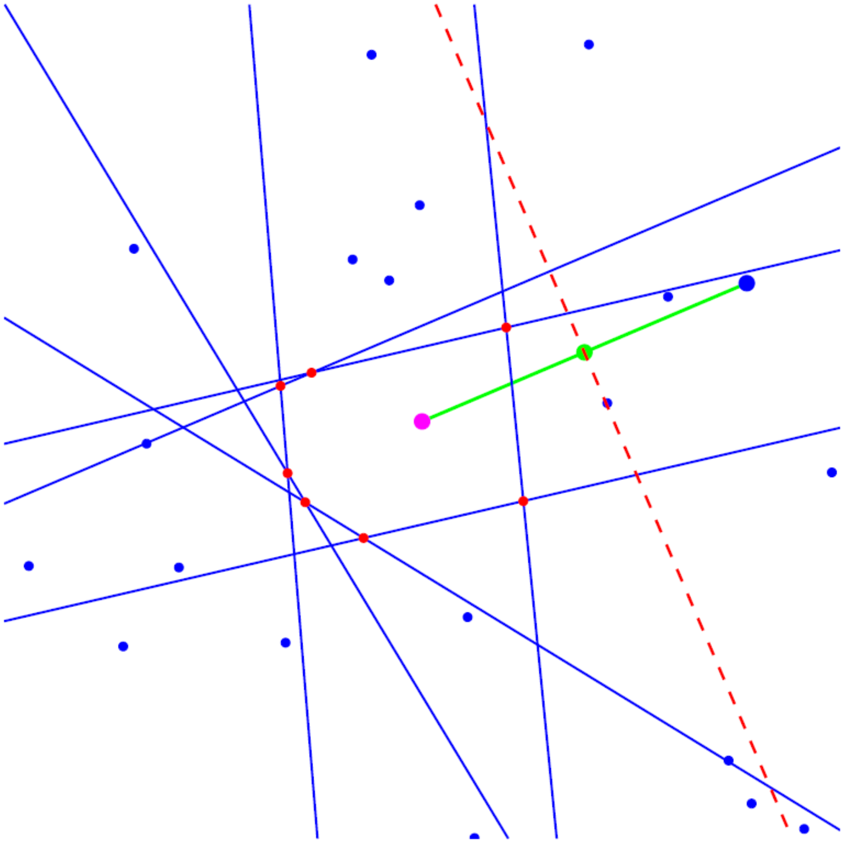

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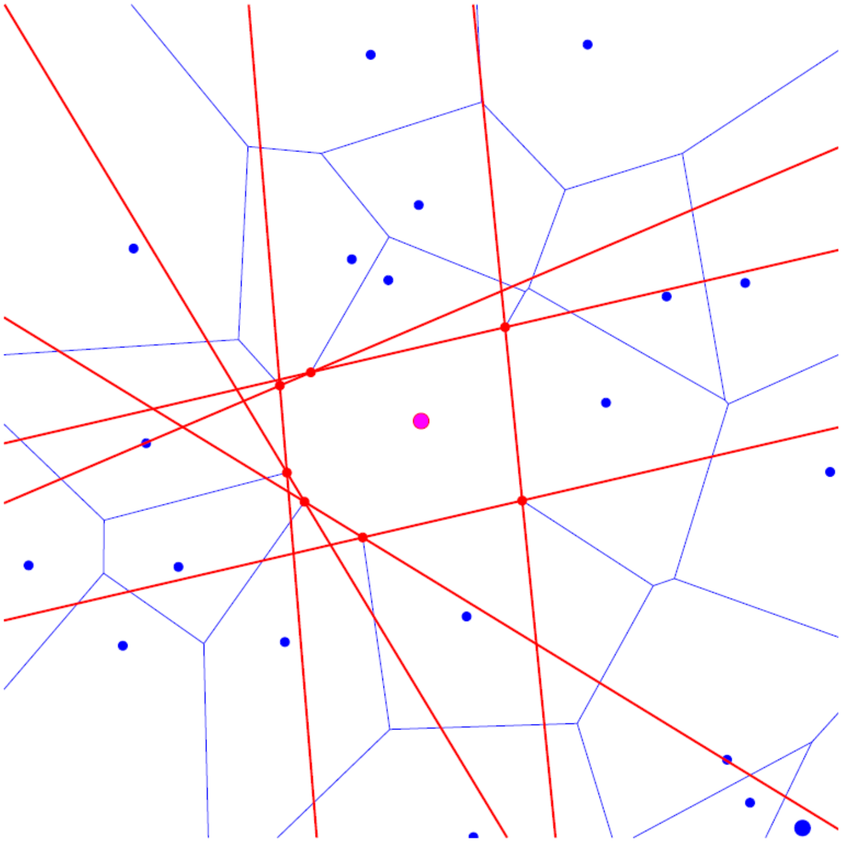
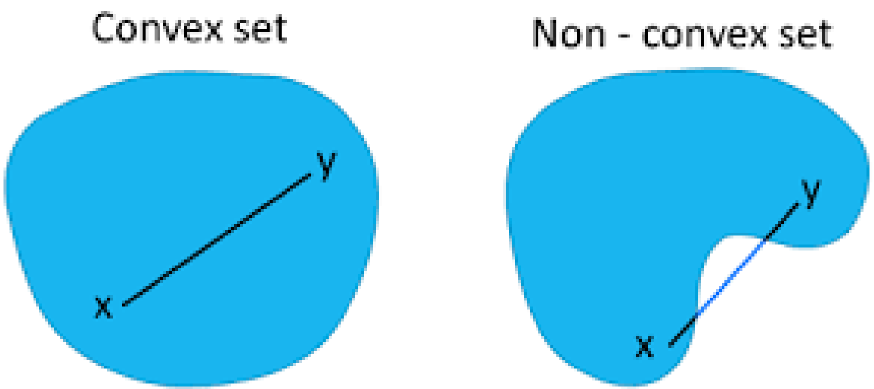


Voronoi Tesselation

Lemma (Properties). Let $\mathcal{V} = \{V_1, \ldots, V_N\}$ denote the Voronoi Tesselation of the point set $P = \{p_1, \ldots, p_N\} \subset \mathbb{R}^d$ in general position. Then the following properties hold true:

(a) Every cell V_i is **convex** (i.e., if $x, y \in V_i$, then $\lambda x + \mu x \in V_i$ for all $\lambda, \mu \in [0, 1]$ with $\lambda + \mu \leq 1$).



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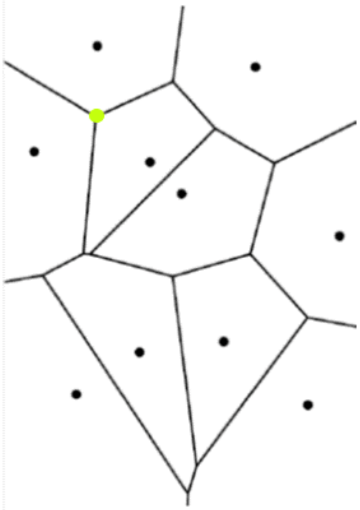
(b) Every vertex of a Voronoi cell V_i lies on the boundary of exactly $d+1$ different cells V_j.



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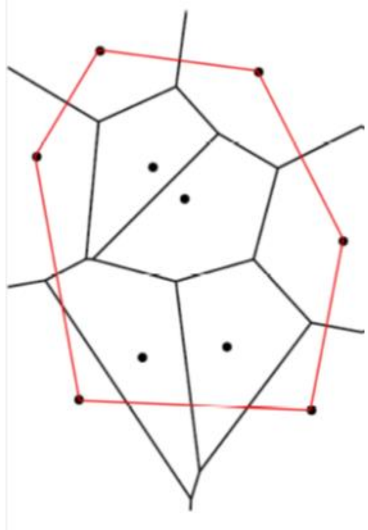
(b) Every vertex of a Voronoi cell V_i lies on the boundary of exactly $d+1$ different cells V_j.

(c) The cell V_i is unbounded if and only if p_i lies on the boundary of the convex hull of P (the convex hull of P is given by $\mathcal{C}(P) = \{\sum_{j=1}^N \lambda_j p_j : \lambda_j \in [0,1], \sum_{j=1}^N \lambda_j \leq 1]\}$).



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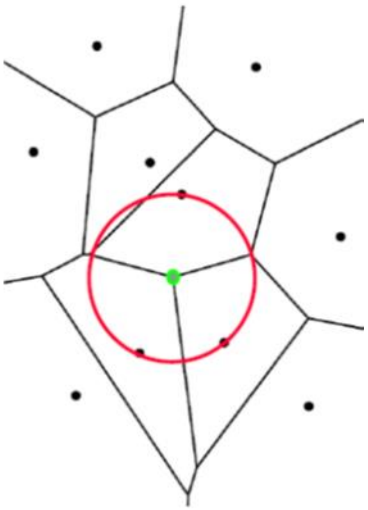
(d) If p is the vertex belonging to the cells $V_{i_1}, \ldots, V_{i_{d+1}}$, then p is the center of the sphere that passes through the points $p_{i_1}, \ldots, p_{i_{d+1}}$. There lie no further points of P within this sphere.



Voronoi Zerlegung – Tabellarische Darstellung 2D

Es muss in irgendeiner Form beschrieben werden, wie die Voronoizellen geometrisch definiert sind:

Tabelle 17: nodes

nodes		
nodeID	**x**	**y**
1	x_1	y_1
⋮	⋮	⋮
max. Anzahl der Nodes n	x_n	y_n

Tabelle 18: voronoiCells

voronoiCells	
cellID	**nodeList**
1	nodeID_1, nodeID_2, nodeID_3, ...
⋮	⋮
max. Anzahl der Zellen m	nodeID_i, nodeID_j, nodeID_k, ...

Tabelle 19: cellCenters

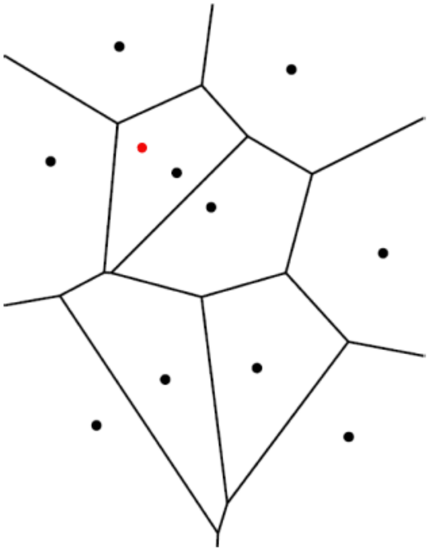
cellCenters		
cellID	**x_center**	**y_center**
1	x_1	y_1
⋮	⋮	⋮
max. Anzahl der zellen m	x_m	y_m

Interpolation mittels Voronoi-Zerlegung: Nearest Neighbor Interpolation

Let $P = \{p_i \in \mathbb{R}^n : i = 1, \dots, N\}$ a set of points in general position (i.e., there are no $n+1$ points of P that lie on an $n-1$-sphere).

A measurement/datum h_i is assigned to every point p_i.

Nearest Neighbor Interpolation: For every $p \in \mathbb{R}^n$ there exists one and only one Voronoi cell V_i with $p \in V_i$. Then, the point p is assigned the value $h(p) = h_i$.



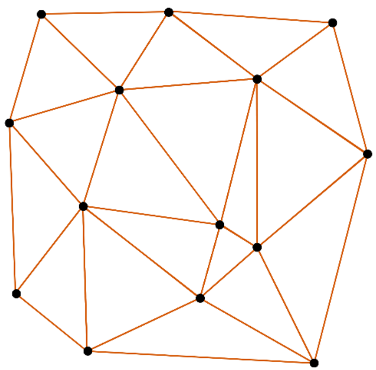
Graphs

A **graph** $\mathcal{G} = (V, E)$ is a pair of abstracte vertices $V = \{v_1, \dots, v_N\}$ and edges $E = \{e_1, \dots, e_M\}$, which each connect two vertices. A graph is called "simple" if

1.) there is at most one edge connecting two vertices,

2.) edges have no orientation,

3.) there exist no loops (i.e., there exists no edge with identical starting and endpoint).



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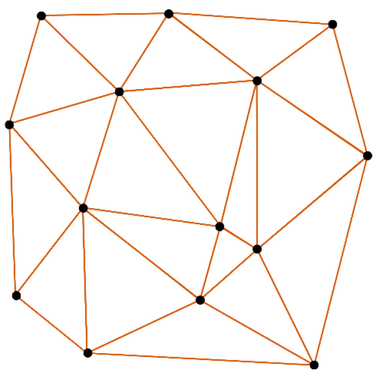
Vertices are **0-cells**, edges **1-cells**, and enclosed faces **2-cells**. Edges determine the "neighborhood" of faces, and vertices determine the "neighborhood" of edges.



Graphs

A graph can be described by its **adjacency matrix** $A_V \in \mathbb{R}^{N \times N}$:

$$(A_V)_{i,j} = \begin{cases} 1, \text{ if } i \neq j \text{ and } v_i, v_j \text{ are connected by an edge} \\ 0, \text{ else} \end{cases}$$

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An equivalent characterization is given by its **incidence matrix** $I_V \in \mathbb{R}^{N \times M}$:

$$(I_{VE})_{i,j} = \begin{cases} 1, \text{ if } v_i \text{ is and endpoint of the edge } e_j \\ 0, \text{ else} \end{cases}$$

Each column of the incidence matrix of a simple graph contains the value 1 exactly twice.

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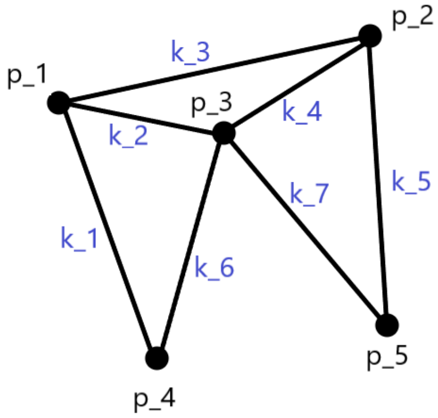

Adjacency matrix of the exemplary graph:

$$A_V = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

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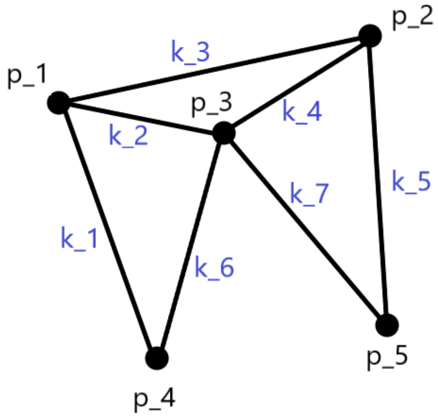

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It holds

$$(A_V)_{i,j} = \begin{cases} (I_{VE} I_{VE}^T)_{i,j}, & \text{if } i \neq j, \\ 0, & \text{else.} \end{cases}$$

Topology of a Voronoi Tesselation

Aside of edges and vertices, a (2-D) Voronoi tesselation contains **faces** $F_1, \ldots, F_K$. The corresponding incidence matrices can be represented as follows:

$$(I_{VE})_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is an endpoint of the edge } e_j \\ 0, & \text{else} \end{cases}$$

$$(I_{EF})_{i,j} = \begin{cases} 1, & \text{if } e_i \text{ is an edge of the boundary of } F_j \\ 0, & \text{else} \end{cases}$$

$$(I_{VF})_{i,j} = (I_{VE} I_{EF})_{i,j} \begin{cases} 1, & \text{if } v_i \text{ is a vertex on the boundary of } F_j \\ 0, & \text{else} \end{cases}$$

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These matrices determine uniquely which vertices, edges, and faces are neighbours. In particular, the adjacency of two Voronoi cells is described by the matrix:

$$(A_F)_{i,j} = \begin{cases} (I_{EF}^T I_{EF})_{i,j}, & \text{if } i \neq j \\ 0, & \text{else.} \end{cases}$$

Dual Graph

The **dual graph** $\mathcal{G}^* = (V^*, E^*)$ with respect to $\mathcal{G}$ is defined by identifying the vertices V^* with the faces F. The set of edges E^* is defined by including an edge between two vertices of V^* if the faces of the original graph were adjacent. In particular, the adjacency matrix A_{V^*} is identical to A_F.

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More general: the dual of a n-dimensional cellular partition is defined by identifying the i-cells with the $(n-i)$-cells of the original partition.

Dual Graph

The **dual graph** $\mathcal{G}^* = (V^*, E^*)$ with respect to $\mathcal{G}$ is defined by identifying the vertices V^* with the faces F. The set of edges E^* is defined by including an edge between two vertices of V^* if the faces of the original graph were adjacent. In particular, the adjacency matrix A_{V^*} is identical to A_F.

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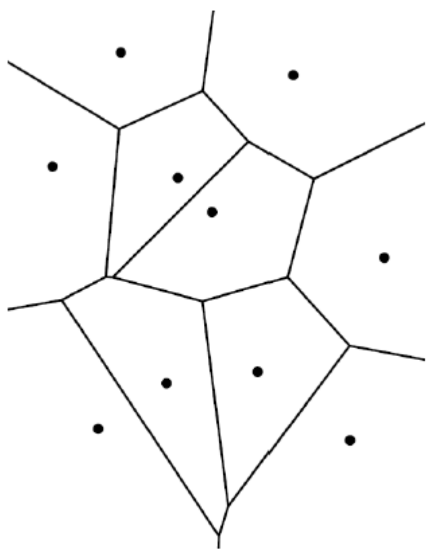
The dual of the Voronoi tesselation yields the **Delaunay triangulation**.



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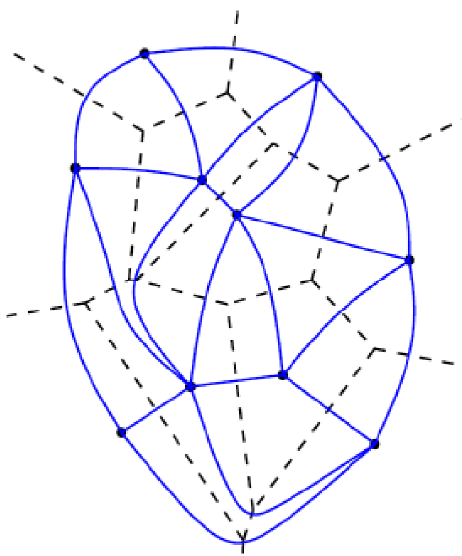

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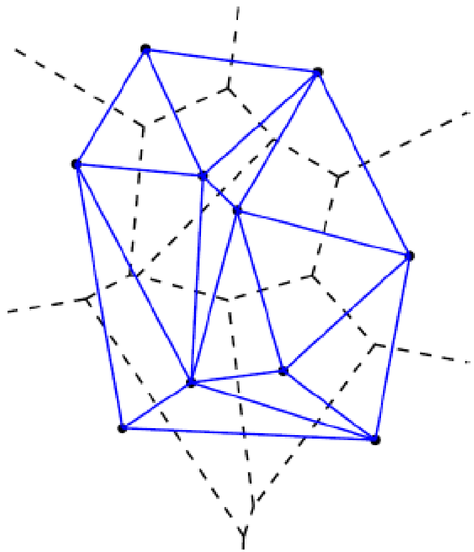
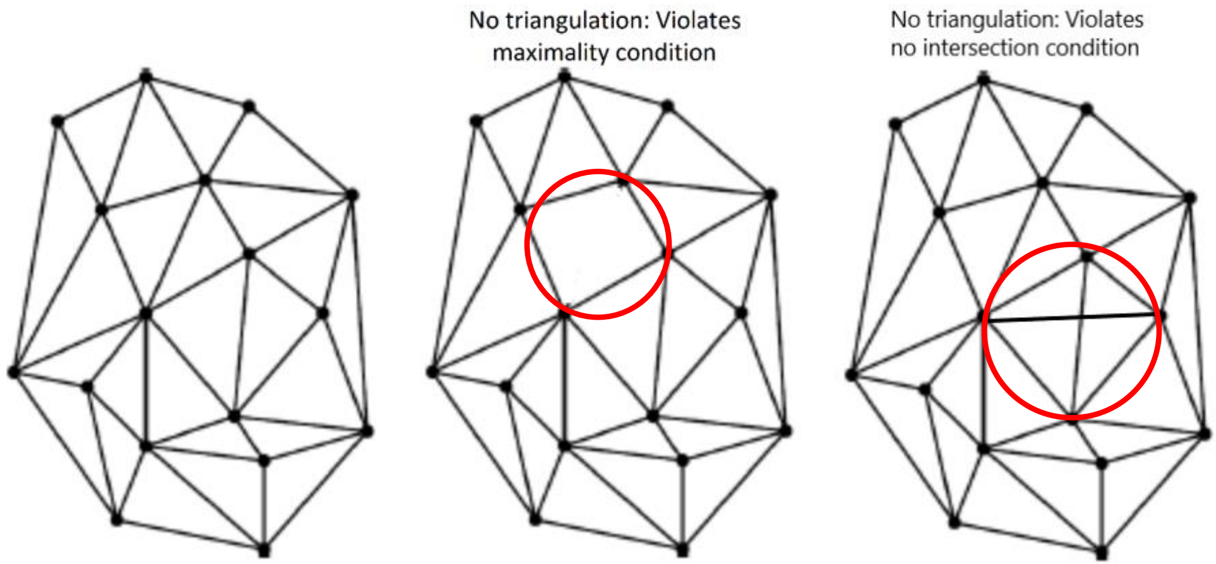


Triangulation

Definition (Triangulation). A triangulation $\mathcal{T}$ of the point cloud $P = \{p_i : i = 1, \ldots, N\} \subset \mathbb{R}^2$ is a maximal family of triangles of the form

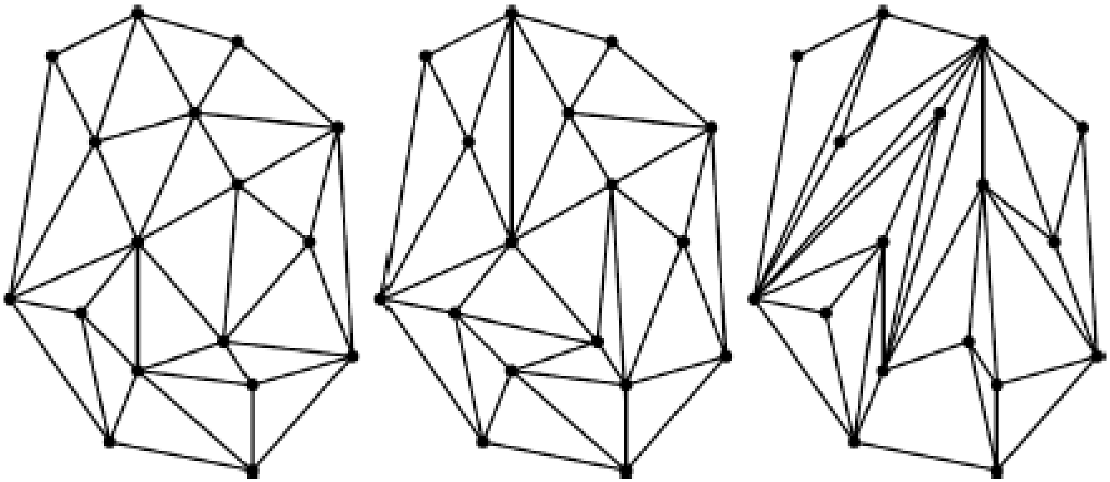
$$\mathcal{T} = \{T_i = \mathcal{C}(p_{i_1}, p_{i_2}, p_{i_3}) : \text{the edges } \overline{p_{i_1}p_{i_2}}, \overline{p_{i_1}p_{i_3}}, \overline{p_{i_2}p_{i_3}} \text{ do not intersect with any edge } \overline{p_{k_1}p_{k_2}}, \overline{p_{k_1}p_{k_3}}, \overline{p_{k_2}p_{k_3}} \text{ of any other triangle } T_k,\ k \neq i,\ i, k = 1, \ldots, M_T\}$$

Maximality is meant in the sense that no triangle could be added to $\mathcal{T}$ without violating the 'no intersection'-property.



Triangulation

Triangulation of a point cloud is in general not unique.



Triangulation

Let $\mathcal{T}$ be a triangulation of the point cloud $P = \{p_i : i = 1, \ldots, N\}$, with all points being in general position. Then the triangulation contains

$$M_T = 2N - K - 2$$

triangles and

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Proof: This follows from Euler's formula for graphs.

The important aspect to remember from this: every triangulation of a given point cloud P has the **same amount of triangles and edges**.

Delaunay Triangulation

Let $\mathcal{D}(\mathcal{V})$ be the dual graph with respect to the Voronoi tesselation $\mathcal{V}$.

Definition (Delaunay Triangulation). A triangulation $\mathcal{T}$ is called a Delaunay triangulation if it is of the form

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Lemma. A Triangulation $\mathcal{T}$ of a pointcloud $P \subset \mathbb{R}^2$ is a Delaunay triangulation if and only if the circumcircle of every triangle $T \in \mathcal{T}$ does not contain any points of P in its interior.